CONTENTS

1. Real Numbers

A Strong Build Foundation

CONCEPT MAP

REAL NUMBERS

1. Euclid's Division Lemma

Given positive integers a and b, there exist unique integers q and r satisfying

$$a = bq + r, 0 \leq r < b$$

2. Euclid's Division Algorithm

Euclid's division algorithm is a technique to compute the Highest Common Factor of two given positive integers.

To get H.C.F. of two positive integers c and d, $c > d$ following steps are to be followed.

(i) Apply Euclid's division lemma to c and d to get whole numbers q and r such that $c = dq + r, 0 \leq r < d$.

(ii) If $r = 0$, then d is H.C.F. of c and d. If $r \neq 0$, apply division lemma to d and r.

(iii) Continue the process till the remainder is zero. The divisor at this stage will be the required H.C.F.

Note:

(i) Euclid's division lemma and algorithm are closely interlinked but lemma is a proven statement used to prove the division algorithm.

(ii) Euclid's division algorithm is stated for only +ve integers but it can be extended for all integers except zero.

3. Fundamental Theorem of Arithmetic

Every composite number can be expressed as product of primes, and this factorisation is unique apart from the order in which the prime factors occur.

Note:

(i) For two positive integers a, b
H.C.F. $(a,b) \times$ L.C.M. $(a,b) = a \times b$

(ii) For three positive integers a, b and c
H.C.F $(a, b, c) \times$ L.C.M. $(a, b, c) \neq a \times b \times c$

(iii) If p is a prime number and it divides a^2, then p also, divides a where 'a' is the positive integer.

4. Irrational Numbers

Number which is not a rational number and whose decimal expansion is non-terminating and non-repeating or

Which cannot be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

5. Rational Numbers and Their Decimal Expansion

(i) If denominator of a rational number is of the form $2^n 5^m$, where n, m are non-negative integers then x has decimal expansion which terminates.

(ii) If decimal expansion of rational number terminates then its denominator has prime factorisation of the form $2^n 5^m$, where n, m are non-negative integers.

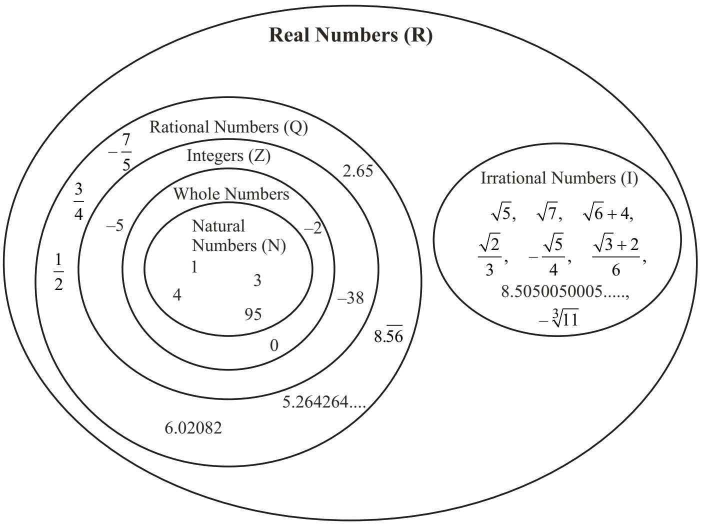
(iii) If denominator of a rational number is not of the form $2^n 5^m$, where n and m are non-negative integers then the rational number has decimal expansion which is non-terminating repeating.

6. *H.C.F and L.C.M of Polynomials

Express each polynomial as a product of irreducible factors with power and take the all common factors with smallest powers as a H.C.F. Take all different factors with greatest power as a L.C.M.

* Belongs to connecting topic.

DIFFERENT TYPES OF REAL NUMBERS



SYMBOLS OF SOME SETS OF SPECIFIC TYPE NUMBERS

R	:	set of all real numbers	R^+	:	set of all positive real numbers
Z	:	set of all integers	Z^+	:	set of all positive integers
Z^-	:	set of all negative integers	Q	:	set of all rational numbers
Q^+	:	set of all positive rational numbers	Q^-	:	set of all negative rational numbers
W	:	set of all whole numbers	N	:	set of all natural numbers

Note : (a) Identity element of addition of a given real number is a real number which when added to the given number, the sum will be the given number itself. Therefore, 0 is the identity element of addition for any real number 'a' because $a + 0 = 0 + a = a$, for all real number a.

(b) Identity element of multiplication of a given real number is a real number which when multiply to the given number, the product will be the given number itself. Therefore, 1 is the identity element of multiplication for any real number 'a', because $1.\ a = a.\ 1 = a$, for all real number a.

(c) Additive inverse of a given real number is a real number which when added to the given number, the sum will be 0 (zero). Therefore, $-a$ is called the additive inverse of any real number a, because $a + (-a) = (-a) + a = 0$, for all real number a.

(d) Multiplicative inverse of a given real number is a real number which when multiply to the given number, the product will become 1. Therefore, $\frac{1}{a}$ is the multiplicative inverse of any non-zero real number a, because $a \times \frac{1}{a} = 1$, for all real number a.

TEST FOR DIVISIBILITY

(i) A number is divisible by 2 when its units digit is even or zero e.g., 350, 7916, etc. 0 is an even number.

(ii) A number is divisible by 3 when the sum of its digits is divisible by 3 e.g., 342, 13791, etc.

(iii) A number is divisible by 4 when the number formed by last two right hand digits is divisible by 4 or if the last two digits are zero. e.g., 1264, 1500, etc.

(iv) A number is divisible by 5 when its unit's digit is either 5 or 0 e.g., 50, 245, etc.

(v) A number is divisible by 6 when it is divisible by 2 and 3 both e.g., 354.

(vi) A number is divisible by 8 when the number formed by the last three right hand digits is divisible by 8 or when the last three digits are zero e.g., 1000, 87895432, etc.

(vii) A number is divisible by 9 when the sum of its digits is divisible by 9 e.g., 39537.

(viii) A number is divisible by 10 when its unit's digit is 0 e.g., 510, 1350, etc.

(ix) A number is divisible by 11 when the difference between the sum of the digits in the odd and even places is either zero or a multiple of 11.

(x) A number is divisible by 12 when it is divisible by 3 and 4 both e.g., 624, etc.

(xi) A number is divisible by 25 when either last two digits are zero or the number formed by last two right hand digits is divisible by 25 e.g., 123475, etc.

(xii) A number is divisible by 125 when the number formed by last three right hand digits is divisible by 125 e.g., 743625, etc.

Did You Know? *The equals sign (=) was invented by Welsh mathematician Robert Recorde in 1557.*

EUCLID'S DIVISION LEMMA

Euclid's Division Lemma states that for given positive integer a and b, there exist unique positive integers q and r satisfying

$a = bq + r$; $0 \le r < b$.

ILLUSTRATION : 1

Find the quotient and remainder q and r for the pairs of positive integers given below :
(i) 23, 4 (ii) 81, 3 (iii) 12, 5

SOLUTION :

(i) 23, 4
$23 = 4 \times 5 + 3$; $q = 5$; $r = 3$ and $0 \le r < 5$
When 23 is divided by 4, quotient is 5 and remainder is 3.

(ii) 81, 3
$81 = 3 \times 27 + 0$
When 81 is divided by 3, quotient is 27 and remainder is 0.
So, $q = 27$; $r = 0$ and $0 \le r < 27$

(iii) 12 , 5
$12 = 5 \times 2 + 2$.
On dividing 12 by 5, we have quotient 2 and remainder 2.
So, $q = 2$; $r = 2$ and $0 \le r < 2$.

Remark : The above Euclid's lemma is nothing but a restatement of the long division process.

TO FIND THE H.C.F. OF TWO POSITIVE INTEGERS USING EUCLID'S DIVISION ALGORITHM

To obtain the H.C.F. of two positive integers, say a and b,

with $a > b$, follow the step below :

Step (i) : Apply Euclid's division lemma, to a and b.
So, we find whole numbers, q and r such that $a = bq + r$, $0 \le r < b$

Step (ii) : If $r = 0$, b is the H.C.F. of a and b.
If $r \ne 0$, apply the division lemma to b and r.

Step (iii) : Continue the process till the remainder become zero.
The divisor at this stage will be the required H.C.F.

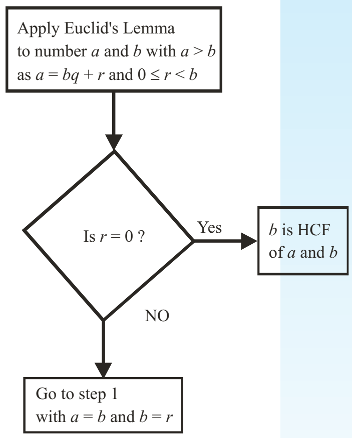


ILLUSTRATION : 2

Let us find the H.C.F. of 60 and 108 using the Euclid's Division Algorithm.

SOLUTION :

Step (i) : Since, $108 > 60$ applying Euclid's Lemma to 60 and 108, we have
$108 = 60 \times 1 + 48$ where $0 \le 48 < 60$

Step (ii) : Since, remainder $48 \ne 0$
So, again applying the division lemma to 60 and 48, we have
$60 = 48 \times 1 + 12$ where $0 \le 12 < 48$.

Step (iii) : Again remainder $12 \ne 0$ so applying division lemma to 48 and 12, we get
$48 = 12 \times 4 + 0$, Here remainder is zero.
Therefore, 12 is the required H.C.F.

ILLUSTRATION : 3

Let us find H.C.F. of three numbers 140, 90 and 54 using the Euclid's Division Algorithm.

SOLUTION :

Since 140 > 90, applying Euclid's Lemma

$$140 = 90 \times 1 + 50$$

Since, remainder $50 \neq 0$

So, again applying Euclid's Lemma to 90 and 50, we have

$$90 = 50 \times 1 + 40$$
$$50 = 40 \times 1 + 10$$
$$40 = 10 \times 4 + 0$$

Here, remainder is zero.

Therefore, H.C.F. of 140 and 90 is 10

Now, find the H.C.F. of 10 and the third number 54.

Since, 54 > 10, then by Euclid's Lemma

$$54 = 10 \times 5 + 4$$
$$10 = 4 \times 2 + 2$$
$$4 = 2 \times 2 + 0$$

Here, remainder is zero. Therefore, 2 is H.C.F. of 10 and 54.

Hence, required H.C.F. (140, 90, 54) = 2.

FUNDAMENTAL THEOREM OF ARITHMETIC

Every composite number can be expressed (factorised) as a product of primes and this factorisation is unique, except the order in which the prime factors occur.

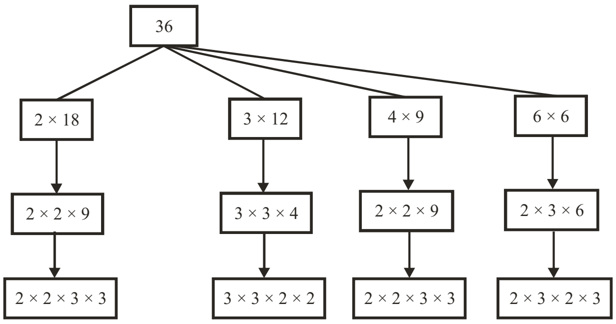
ILLUSTRATION : 4

Express the following numbers as the product of prime factors.

(i) 36 **(ii) 156**

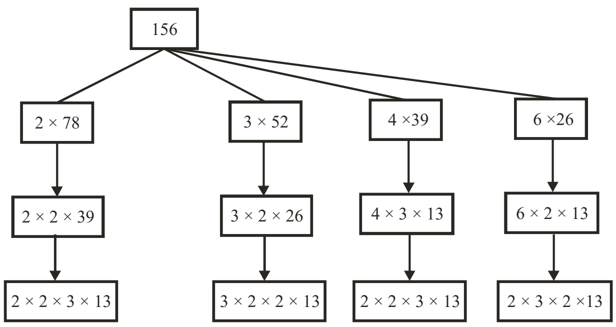
SOLUTION :

(i) 36



In each of the cases, product of prime factors of 36 is $2 \times 2 \times 3 \times 3$.

(ii) 156



So, in each of the cases prime factors of 156 is $2 \times 2 \times 3 \times 13$.

Hence, we can conclude that the prime factorisation of a number is unique.

TO FIND THE H.C.F. AND L.C.M. BY PRIME FACTORISATION METHOD

Algorithm

(i) Factorize and express integers as a product of primes with powers.
(ii) To find H.C.F. find the product of each common prime factor(s) with smallest power involved in the numbers.
(iii) To find L.C.M. find the product of each prime factors with greatest power involved in the numbers.
(iv) For any two positive numbers a and b, H.C.F. $(a, b) \times$ L.C.M. $(a, b) = a \times b$

Learn More

Let $\frac{a}{b}, \frac{p}{q}$ and $\frac{\ell}{m}$ be some fractions, then

(a) LCM of $\frac{a}{b}, \frac{p}{q}$ and $\frac{\ell}{m} = \frac{\text{LCM}(a,p,\ell)}{\text{HCF}(b,q,m)}$

(b) HCF of $\frac{a}{b}, \frac{p}{q}$ and $\frac{\ell}{m} = \frac{\text{HCF}(a,p,\ell)}{\text{LCM}(b,q,m)}$

ILLUSTRATION : 5

Find the L.C.M. and H.C.F. of the following integers by applying the prime factorisation method. 8, 9 and 25

SOLUTION :

First we write the prime factorisation of each of the given numbers.
$8 = 2 \times 2 \times 2 = 2^3$, $9 = 3 \times 3 = 3^2$, $25 = 5 \times 5 = 5^2$
$\therefore$ L.C.M. $= 2^3 \times 3^2 \times 5^2 = 8 \times 9 \times 25 = 1800$ and H.C.F. = 1

Remark : The product of two positive integers is equal to the product of their H.C.F. and L.C.M., but the same is not true for three or more positive integers.

Did You Know?

The symbol π is called pi and this is a famous irrational number. The symbol π was introduced by an Anglo-Welsh philogist William Jones in 1706.

CONSEQUENCE OF FUNDAMENTAL THEOREM

Let p be a prime number and 'a' be a positive integer. If p divides a^2, then p divides 'a'.

PROVING IRRATIONALITY OF NUMBERS

In proving a number is an irrational number or not, we will use the 'consequence of fundamental theorem of Arithmetic'.

ILLUSTRATION : 6

Prove that $\sqrt{2}$ is irrational.

SOLUTION :

Let us assume, to the contrary, that $\sqrt{2}$ is rational. So, we can find integers r and s $(\neq 0)$ such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are co-prime i.e. they have no common.
Factor other than 1. So, $b\sqrt{2} = a$
Squaring on both sides and rearranging, we get $2b^2 = a^2$. Here we see that, 2 divides a^2. Since 2 is a prime number and divides square of a positive number 'a', therefore according to consequence of fundamental theorem 2 divides a also. So, we can write $a = 2c$ for some integer c.
Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.
This means that 2 divides b^2, and so 2 divides b (again using consequence of fundamental theorem)
Therefore, a and b have at least 2 as a common factor.
But this contradicts the fact that a and b have no common factors other than 1.
This contradiction, has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.
So, we conclude that $\sqrt{2}$ is irrational.

Learn More

Some properties of Rational and Irrational numbers.

(a) Sum or difference of two or more irrational number is either zero or an irrational number

(i) $\sqrt{7}+(-\sqrt{7})=0$ (ii) $\sqrt{3}-\sqrt{3}=0$

(iii) $2\sqrt{5}-\sqrt{5}=\sqrt{5}$ (irrational) (iv) $\sqrt{2}-8\sqrt{2}=-7\sqrt{2}$ (irrational)

(v) $\sqrt{3}+\sqrt{5}-\sqrt{3}=\sqrt{5}$ (irrational)

(b) Product of two or more irrational numbers can be rational or irrational

(i) $\sqrt{3}\times\sqrt{3}=3$ (rational) (ii) $5^{\frac{1}{3}}\cdot5^{\frac{1}{3}}\cdot5^{\frac{1}{3}}=5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=5^{\frac{3}{3}}=5$ (rational)

(iii) $\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}=2\sqrt{2}$ (irrational) (iv) $\sqrt{8}\cdot\sqrt{7}=\sqrt{56}=2\sqrt{14}$ (irrational)

(c) Division of one irrational number by another irrational number can be rational or irrational .

(i) $\dfrac{2\sqrt{7}}{\sqrt{7}}=2$ (rational) (ii) $\dfrac{5\sqrt{3}}{\sqrt{2}}=5\dfrac{\sqrt{3}}{2}$ (irrational)

DECIMAL EXPANSION OF RATIONAL NUMBERS

(i) If a number in the form $\frac{p}{q}$ is such that p and q are co-prime (i.e. p and q have no common factor other than 1) and prime factorisation of q is of the form $2^n\,5^m$, where n and m are non-negative integers then decimal expansion of $\frac{p}{q}$ terminates i.e. when we divide p by q, then we get an integer or a number upto a certain decimal place.

(ii) Let x be a rational number whose decimal expansion terminates. Then, x can be expressed in the form $\frac{p}{q}$ where p and q are co-prime, and the prime factorisation of q is of the form $2^n\,5^m$; where n, m are non-negative integers.

(iii) Let $x=\frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form $2^n 5^m$; where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring) i.e. decimal expansion of x has infinite digits after decimal point but no part of digits after decimal point repeat continuously.

ILLUSTRATION : 7

Without performing the long division, find the decimal expansion of the following expressions.

(i) $\mathbf{\dfrac{3}{8}}$ **(ii)** $\mathbf{\dfrac{13}{125}}$ **(iii)** $\mathbf{\dfrac{7}{80}}$ **(iv)** $\mathbf{\dfrac{14588}{625}}$

SOLUTION :

(i) $\dfrac{3}{8}=\dfrac{3}{2^3}=\dfrac{3\times5^3}{2^3\times5^3}=\dfrac{375}{10^3}=0.375$

(ii) $\dfrac{13}{125}=\dfrac{13}{5^3}=\dfrac{13\times2^3}{2^3\times5^3}=\dfrac{104}{10^3}=0.104$

(iii) $\dfrac{7}{80}=\dfrac{7}{2^4\times5}=\dfrac{7\times5^3}{2^4\times5^4}=\dfrac{875}{10^4}=0.0875$

(iv) $\dfrac{14588}{625}=\dfrac{2^2\times7\times521}{5^4}=\dfrac{2^6\times7\times521}{2^4\times5^4}=\dfrac{233408}{10^4}=23.3408$

ILLUSTRATION : 8

Convert 6.235 in the form of $\mathbf{\frac{p}{q}}$, where p and q are co-prime. Also show that q is in the form $2^n\times5^m$ are non-negative integers.

SOLUTION :

$6.235=\dfrac{6235}{1000}=\dfrac{1247}{200}$, which is in the form of $\frac{p}{q}$. Also $q=200=2^3\times5^2$, which is in the form $2^n\times5^m$.

Note: We can also convert a rational number which is non-terminating reccuring (i.e., repeating) in the form of $\frac{p}{q}$, where p and q are co-prime (but $q\neq0$).

ILLUSTRATION : 9

Convert $23.\overline{426}$ into $\frac{p}{q}$ form, where p and q are coprime (but $q \neq 0$).

SOLUTION :

Let $x = 23.\overline{426} \Rightarrow x = 23.426\ 426\ 426 .....$ ... (i)

Multiply equation (i) by 1000, so that in the right hand side of the equation (i), the point (.) comes just after the first repeating part (426).

$\therefore\ 1000x = 23\,426.426\ 426 ....$ ... (ii)

Subtracting equation (i) from (ii), we get $999x = 23403$

$\therefore\quad x = \frac{23403}{999} = \frac{7801}{333}\quad \therefore\ 23.\overline{426} = \frac{7801}{333}$, which is in the required form of $\frac{p}{q}$.

Time to Check Your Knowledge

Show that there is no positive integer n for which $\sqrt{n-1}+\sqrt{n+1}$ is rational.

SOLUTION

We assume that there is a positive integer n for which $\sqrt{n-1}+\sqrt{n+1}$ is rational and equal to $\frac{a}{b}$, where a and b are positive integers.

So, $\sqrt{n-1}+\sqrt{n+1} = \frac{a}{b}$ ...(i) $\Rightarrow \frac{b}{a} = \frac{1}{\sqrt{n-1}+\sqrt{n+1}}$,

Rationalising RHS by multiplying numerator and denominator by $(\sqrt{n-1}-\sqrt{n+1})$, we get

$$\frac{b}{a} = \frac{(\sqrt{n-1}-\sqrt{n+1})}{(\sqrt{n-1}+\sqrt{n+1})(\sqrt{n-1}-\sqrt{n+1})} = \frac{\sqrt{n-1}-\sqrt{n+1}}{(n-1)-(n+1)} = \frac{\sqrt{n-1}-\sqrt{n+1}}{-2}$$

or, $\frac{b}{a} = \frac{\sqrt{n+1}-\sqrt{n-1}}{2} \Rightarrow \sqrt{n+1}-\sqrt{n-1} = \frac{2b}{a}$ ...(ii)

Adding equations (i) and (ii), we get

$2\sqrt{n+1} = \frac{a}{b}+\frac{2b}{a} = \frac{a^2+2b^2}{ab} \Rightarrow \sqrt{n+1} = \frac{a^2+2b^2}{2ab}$ ...(iii)

Subtracting eqn. (ii) from (i), we get, $2\sqrt{n-1} = \frac{a}{b}-\frac{2b}{a} = \frac{a^2-2b^2}{ab} \Rightarrow \sqrt{n-1} = \frac{a^2-2b^2}{2ab}$ ...(iv)

From (iii) and (iv), we have $\sqrt{n+1}$ and $\sqrt{n-1}$ are both rational as a and b are both positive integer.

This is possible only when both $n+1$ and $n-1$ are perfect squares of some positive integer n.

$(n-1)$ and $(n+1)$ differ by 2. But two perfect squares never differ by 2, hence $(n-1)$ and $(n+1)$ both cannot be perfect square. So, there is no positive number n for which $\sqrt{n-1}+\sqrt{n+1}$ is a rational number.

CONNECTING TOPIC

H.C.F. of polynomials

To find H.C.F. of two or more given polynomials follow the following steps :

Step (i) : Express each polynomial as a product of irreducible factors (simple factors) with power. Numerical factors, if any, are expressed as product of primes with power.

Step (ii) : If there is no common factor, then the H.C.F. is 1. If there are common simple factors find the smallest powers of these irreducible factors in the factorised form of the polynomials.

Step (iii) : Raise the irreducible factors to the smallest power found in step (ii) and multiply them to get the H.C.F.

L.C.M. of polynomials

To find L.C.M. of polynomial follow the following steps :

Step (i) : Express each polynomial as a product of irreducible factors with power. Express numerical factors, if any, as product of primes with power.

Step (ii) : Consider all the irreducible factors occurring in the given polynomials once only. Find the greatest power of each of these irreducible factors in the factorised form of the given polynomials.

Step (iii) : Raise each irreducible factor to the greatest power found in step (ii), and multiply them to get the L.C.M.

CONNECTING TOPIC CONNECTING TOPIC

CASE STUDY : Euclid's Division Lemma, L.C.M. and H.C.F.

- **Euclid's Division Lemma**
  Given positive integers a and b, there exist unique whole numbers q and r satisfying $a = bq + r$, $0 \le r < b$.
  Positive integer in different form.
  $a = bq + r$ form is nothing but a division algorithm, where 'a' is divided by 'b', then quotient 'q' and remainder 'r' is obtained.
  If a number a in the form $bq + r$ i.e. $a = bq + r$, $0 \le r < b$ represent all positive integer for different value of b and q.
- In practical application questions of L.C.M. and H.C.F., students face difficulty whether they take L.C.M. or H.C.F. in a given problem.
  If minimum word is used in the problem, then use L.C.M. and if maximum word is used in the problem, then use H.C.F.
  Let us understand the above concepts in a better way in different cases through illustrative examples.

CASE-I : Positive integer in different form.
Illustration: *Show that any positive integer is of the form $3q$ or, $3q + 1$ or, $3q + 2$ for some integer q.*
Sol. Let a be any positive integer and $b = 3$. Applying division Lemma with a and $b = 3$, we have
$a = 3q + r$, where $0 \le r < 3$ and q is some integer
$\Rightarrow$ $a = 3q + 0$ or, $a = 3q + 1$ or, $a = 3q + 2$
$\Rightarrow$ $a = 3q$ or, $a = 3q + 1$ or, $a = 3q + 2$ for some integer q.

CASE-II : Use of L.C.M.
Illustration: *On a morning walk, three persons step off together and their steps measure 40 cm, 42 cm and 45 cm, respectively. What is the minimum distance each should walk so that each can cover the same distance in complete steps?*
Sol. Step measures of three persons are 40 cm, 42 cm and 45 cm.
The minimum distance each should walk so that each can cover the same distance in complete steps is the L.C.M. of 40 cm, 42 cm and 45 cm.
Prime factorisation of 40, 42 and 45 gives
$40 = 2^3 \times 5$, $42 = 2 \times 3 \times 7$, $45 = 3^2 \times 5$
L.C.M. (40, 42, 45) = Product of the greatest power of each prime factor involved in the numbers $= 2^3 \times 3^2 \times 5 \times 7 = 8 \times 9 \times 35 = 72 \times 35 = 2520$ cm.

CASE-III : Use of H.C.F.
Illustration: *The areas of three fields are 165 m^2, 195 m^2 and 285 m^2 respectively. From these, flowers beds of equal size are to be made. If the breadth of each bed be 3 metres, what will be the maximum length of each bed?*
Sol. The areas of three fields are 165 m^2, 195 m^2 and 285 m^2.
Maximum area of a flower bed = H.C.F. of 165 and 195 and 285.
We first find the H.C.F. of 165 and 195.
Using Euclid's Algorithm, we have the following equations:
$195 = 165 \times 1 + 30$
$165 = 30 \times 5 + 15$
$30 = 15 \times 2 + 0$
Now, the remainder is zero. Since the divisor at this stage is 15.
$\therefore$ H.C.F. (165, 195) = 15. Now we find the H.C.F. of 15 and 285.
Using Euclid's Algorithm, we have the following equation:
$285 = 15 \times 17 + 0$
Now, the remainder is zero. Since the divisor at this stage is 15.
$\therefore$ H.C.F. (15, 285) = 15. So H.C.F. of 165, 195 and 285 = 15
$\therefore$ Length of a flower bed $= \frac{15}{3} = 5$m.

Think Out of the Box

Q. 1. The traffic lights at three different road crossing change after 60 sec, 75 sec and 90 sec respectively. If they all change simultaneously at 9 : 30 A.M., when will they again change simultaneously.
[**Hint:** Find L.C.M. of 60, 75 and 90. **Ans.** 9 : 45 A.M.]

Q. 2. The floor of room has dimensions 12 m and 10 m can be covered completely by square tiles. Find least number of square tiles.
[**Hint:** No. of tiles = Area of floor / Area of tile **Ans.** 30]

Q. 3. A number when divided by 187 leaves remainder 35, what will be the remainder when the same number is divided by 17?
[**Hint:** Divide 187 and 35 by 17 and add remainder **Ans.** 1]

MISCELLANEOUS SOLVED EXAMPLES

1. If d is the H.C.F. of 56 and 72, find Integral values of x, y satisfying $d = 56x + 72y$.

Sol. Applying Euclid's division lemma to 56 and 72, we get

$72 = 56 \times 1 + 16$ ..... (i) $\left[\because 56\overline{)72}(1 \quad \frac{56}{16}\right]$

Since the remainder $16 \neq 0$, so, we consider the divisor 56 as dividend and the remainder 16 as divisor. Now, on applying division lemma, we get

$56 = 16 \times 3 + 8$ ..... (ii) $\left[\because 16\overline{)56}(3 \quad \frac{48}{8}\right]$

Since the remainder $8 \neq 0$, so, we consider the divisor 16 as dividend and the remainder 8 as divisor. Now, on applying division algorithm, we get

$16 = 8 \times 2 + 0$ .... (iii) $\left[\because 8\overline{)16}(2 \quad \frac{16}{0}\right]$

We observe that the remainder at this stage is zero. Therefore, last divisor 8 is the H.C.F. of 56 and 72.

From (ii), we get, $8 = 56 - 16 \times 3 \Rightarrow 8 = 56 - (72 - 56 \times 1) \times 3$ [$\because 16 = 72 - 56 \times 1$ (from (i))]

$\Rightarrow 8 = 56 - 3 \times 72 + 56 \times 3 \Rightarrow 8 = 56 \times 4 + (-3) \times 72 \quad \therefore \quad x = 4$ and $y = -3$.

2. Show that any positive odd integer is of the form $8q + 1$, $8q + 3$, $8q + 5$, $8q + 7$, where q is some integer.

Sol. Let a and ($b = 8$) be two positive integers where a is odd.

Applying division lemma $a = 8q + r$ where $0 \leq r < 8$. So, r can take any of the values 0, 1, 2, 3, 4, 5, 6, 7

Therefore, $a = 8q$, $8q + 1$, $8q + 2$, $8q + 3$, $8q + 4$, $8q + 5$, $8q + 6$, $8q + 7$, $8q + 8$

Since, a is odd. Therefore, a cannot take values $8q$, $8q + 2$, $8q + 4$, $8q + 8$, since they can expressed as multiples of 2.

So, a will take values $8q + 1$, $8q + 3$, $8q + 5$ and $8q + 7$.

3. Write the decimal expansion of the following using prime factorisation :

(i) $\frac{35}{16}$ **(ii)** $\frac{17}{8}$ **(iii)** $\frac{327}{500}$

Sol. (i) $\frac{35}{16} = \frac{35 \times 5^4}{2^4 \times 5^4} = \frac{35 \times 625}{(10)^4} = \frac{21875}{10000} = 2.1875$ (ii) $\frac{17}{8} = \frac{17 \times 5^3}{2^3 \times 5^3} = \frac{17 \times 125}{(10)^3} = \frac{2125}{1000} = 2.125$

(iii) $\frac{327}{500} = \frac{327}{5 \times 5 \times 5 \times 2 \times 2} = \frac{327}{5^3 \times 2^2} = \frac{327 \times 2}{5^3 \times 2^3} = \frac{654}{(10)^3} = 0.654$

4. Prove that if x and y are odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.

Sol. We know that any odd positive integer is of the form $2q + 1$ for some integer q.

So, let $x = 2m + 1$ and $y = 2n + 1$ for some integers m and n.

$\therefore \quad x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2 \Rightarrow x^2 + y^2 = 4(m^2 + n^2) + 4(m + n) + 2$

$\Rightarrow x^2 + y^2 = 4q + 2$, where $q = (m^2 + n^2) + (m + n) \Rightarrow x^2 + y^2$ is even and leaves remainder 2 when divided by 4

$\Rightarrow x^2 + y^2$ is even but not divisible by 4

5. Two bills of ₹ 6075 and ₹ 8505 respectively are to be paid separately by cheques of same amount. Find the largest possible amount of each cheque.

Sol. Largest possible amount of cheque will be the H.C.F. of 6075 and 8505.

Applying Euclid's division lemma to 8505 and 6075, we have, $8505 = 6075 \times 1 + 2430$

Since, remainder $2430 \neq 0$, therefore again apply division lemma to 6075 and 2430

$6075 = 2430 \times 2 + 1215$. Again remainder $1215 \neq 0$

So, again apply division lemma to 2430 and 1215

$2430 = 1215 \times 2 + 0$. Here the remainder is zero. So, H.C.F. = 1215

Therefore, the largest possible amount of each cheque will be 1215.

6. Three wheels can complete respectively 60, 36, 24 revolutions per minute. There is a red spot on each wheel that touches the ground at time zero. After how much time, all these spots will simultaneously touch the ground again?

Sol. 1^{st} wheel makes 1 revolutions per sec; 2^{nd} wheel makes $\frac{6}{10}$ revolutions per sec. 3^{rd} wheel makes $\frac{4}{10}$ revolutions per sec

In other words 1^{st}, 2^{nd} and 3^{rd} wheel take 1, $\frac{5}{3}$ and $\frac{5}{2}$ seconds respectively to complete one revolution.

L.C.M. of 1, $\frac{5}{3}$ and $\frac{5}{2} = \frac{\text{L.C.M. of } 1, 5, 5}{\text{H.C.F. of } 1, 3, 2} = 5$

Hence, after every 5 seconds the red spots on all the three wheels touch the ground.

7. Find the largest number that will divide 398, 436 and 542 leaving remainders 7, 11 and 15 respectively.

Sol. Clearly, the required number is the H.C.F. of the numbers

$398 - 7 = 391$, $436 - 11 = 425$ and $542 - 15 = 527$.

First we find the H.C.F. of 391 and 425 by Euclid's algorithm as given below :

$425 = 391 \times 1 + 34$; $391 = 34 \times 11 + 17$; $34 = 17 \times 2 + 0$

Clearly, H.C.F. of 391 and 425 is 17.

Let us now find the H.C.F. of 17 and the third number 527 by Euclid's algorithm:

$527 = 17 \times 31 + 0$

The H.C.F. of 17 and 527 is 17. Hence, H.C.F. of 391, 425 and 527 is 17. Hence, the required number is 17.

SOLVED EXAMPLES BASED ON CONNECTING TOPIC

8. If p is a prime number greater than 3, then find the number by which $(p^2 - 1)$ is always divisible :

Sol. Let $p = 5, 7, 11, 13, ...$

For $p = 5$, $(p^2 - 1) = 24$; For $p = 7$, $(p^2 - 1) = 48$

For $p = 11$, $(p^2 - 1) = 120$; For $p = 13$, $(p^2 - 1) = 168$

....

....

Clearly, all the above numbers 24, 48, 120 and 168 are divisible by 24.

9. What is the value of M and N if M39048458N is divisible by 8 and 11, where M and N are single digit integers?

Sol. A number is divisible by 8 if the number formed by the last three digits is divisible by 8.

i.e., $58N$ is divisible by 8. Clearly, $N = 4$

Again, a number is divisible by 11 if the difference between the sum of digits at even places and sum of digits at the odd places is either 0 or is divisible by 11.

i.e., $(M + 9 + 4 + 4 + 8) - (3 + 0 + 8 + 5 + N) = M + 25 - (16 + N) = M - N + 9$, must be zero or it must be divisible by 11.

i.e. $M - N = 11 - 9 \Rightarrow M = 2 + 4 = 6$. Hence, $M = 6$, $N = 4$

10. Find the L.C.M of the polynomials $12(x^4 - x^3)$ and $8(x^4 - 3x^3 + 2x^2)$

Sol. Let the given polynomials be denoted as $p(x)$ and $q(x)$ respectively

We have $p(x) = 2^2 \times 3 \times x^3 \times (x - 1)$; $q(x) = 2^3 \times x^2 \times (x - 1) \times (x - 2)$

Irreducible factors are 2, 3, x, $x - 1$ and $x - 2$. The respective highest exponents are 3, 1, 3, 1 and 1.

$\therefore$ L.C.M. $= 2^3 \times 3 \times x^3 \times (x - 1) \times (x - 2) = 24x^3 (x - 1)(x - 2)$.

11. If a number 774958A96B is to be divisible by 8 and 9, then find the values of A and B.

Sol. According to the question, the number 774958A96B is divisible by 8 and 9. For the number to be divisible by 8, its last three digits have to be divisible by 8. Then 96$\underline{0}$ and 96$\underline{8}$ can be the possibilities. Therefore, B should be either 0 or 8.

For the number to be divisible by 9, the sum of the digits of the number should be divisible by 9.

If B = 0, then the sum of the digits of 774958A96B = A + 55.

Now, 774958A96B to be divisible by 9, (A + 55) should be divisible by 9. It is possible only when $\boxed{A = 8}$.

If B = 8, then the sum of the digits of 774958A96B = A + 63. Now, 774958A968 to be divisible by 9, (A + 63) should be divisible by 9. It is possible only when $\boxed{A = 9}$. Hence, possible value of A and B are (8, 0) or (9, 8).

12. Find the H.C.F. of the polynomials. $150(6x^2 + x - 1)(x - 3)^3$ and $84(x - 3)^2(8x^2 + 14x + 5)$

Sol. Let $f(x) = 150(6x^2 + x - 1)(x - 3)^3$ and $g(x) = 84(x - 3)^2(8x^2 + 14x + 5)$

Now, $f(x) = 150(6x^2 + x - 1)(x - 3)^3 = 150(6x^2 + 3x - 2x - 1)(x - 3)^3 = 150[(6x^2 + 3x) - 1(2x + 1)](x - 3)^3$

$= 150[3x(2x + 1) - 1(2x + 1)](x - 3)^3 = 150(2x + 1)(3x - 1)(x - 3)^3 = 2 \times 3 \times 5^2 (2x + 1)(3x - 1)(x - 3)^3$

$g(x) = 84(x - 3)^2(8x^2 + 14x + 5) = 84(x - 3)^2(8x^2 + 10x + 4x + 5)$

$= 84(x - 3)^2[2x(4x + 5) + 1(4x + 5)] = 84(x - 3)^2(4x + 5)(2x + 1) = 2^2 \times 3 \times 7(x - 3)^2(2x + 1)(4x + 5)$

Hence, required H.C.F. $= 2^1 . 3^1 (2x + 1)^1 . (x - 3)^2 = 6(2x + 1)(x - 3)^2$

Exercise 1 Master Boards

Multiple Choice Questions

DIRECTIONS : *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct.*

1. What is the largest number that divides 70 and 125, leaving remainders 5 and 8 respectively?
(a) 13 (b) 9 (c) 3 (d) 585

2. What is the largest number that divides 245 and 1029, leaving remainder 5 in each case?
(a) 15 (b) 16 (c) 9 (d) 5

3. A class of 20 boys and 15 girls is divided into n groups so that each group has x boys and y girls. Values of x, y and n respectively are
(a) 3, 4 and 8 (b) 4, 3 and 6
(c) 4, 3 and 7 (d) 7, 4 and 3

4. If p, q are two consecutive natural numbers, then H.C.F. (p, q) is
(a) p (b) q (c) 1 (d) pq

5. Given that L.C.M. (91, 26) = 182, then H.C.F. (91, 26) is
(a) 13 (b) 26 (c) 17 (d) 9

6. By Euclid's division lemma $x = qy + r$, $x > y$, the value of q and r for $x = 27$ and $y = 5$ are :
(a) $q = 5, r = 3$ (b) $q = 6, r = 3$
(c) $q = 5, r = 2$ (d) cannot be determined

7. Which of the following statement is true?
(a) Every point on the number line represents a rational number.
(b) Irrational numbers cannot be represented by points on the number line.
(c) $\frac{22}{7}$ is a rational number.
(d) None of these.

8. The sum of exponents of prime factors in the prime-factorisation of 196 is
(a) 3 (b) 4 (c) 5 (d) 2

9. Euclid's division Lemma states that for two positive integers a and b, there exists unique integer q and r satisfying $a = bq + r$, and
(a) $0 < r < b$ (b) $0 < r \le b$
(c) $0 \le r < b$ (d) $0 \le r \le b$

Fill in the Blanks

DIRECTIONS : *Complete the following statements with an appropriate word / term to be filled in the blank space(s).*

1. $\sqrt{5}$ is a/ an .... number.

2. $\frac{1}{\sqrt{2}}$ is a/ an .... number.

3. The exponent of 2 in the prime factorisation of 144, is ....

4. $7\sqrt{5}$ is a/ an .... number.

5. $6+\sqrt{2}$ is a/ an .... number.

6. An .... is a series of well defined steps which gives a procedure for solving a type of problem.

7. An........... is a proven statement used for proving another statement.

8. L.C.M. of 96 and 404 is ....

9. H.C.F. of 6, 72 and 120 is ....

10. 156 as a product of its prime factors ....

11. $\frac{35}{50}$ is a .... decimal expansion.

True / False

DIRECTIONS : *Read the following statements and write your answer as true or false.*

1. Given positive integers a and b, there exist whole numbers q and r satisfying $a = bq + r$, $0 \le r < b$.

2. Every composite number can be expressed (factorised) as a product of primes and this factorisation is unique, apart from the order in which the prime factors occur.

3. $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

4. If $x = p/q$ be a rational number, such that the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which is terminating.

5. Any positive odd integer is of the form $6q + 1$ or $6q + 3$ or $6q + 5$, where q is some integer.

6. The quotient of two integers is always a rational number.

7. 1/0 is not rational.

8. The number of irrational numbers between 15 and 18 is infinite.

9. Every fraction is a rational number.

Match the Following

DIRECTIONS : *Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in column-I have to be matched with statements (p, q, r, s) in column-II.*

1.

Column-I	Column-II
(A) Irrational number is always	(p) rational number
(B) Rational number is always	(q) irrational number
(C) $\sqrt[3]{6}$ is not a	(r) non-terminating, non-repeating
(D) $2+\sqrt{2}$ is an	(s) terminating decimal

2. | Column-I | Column-II |
|---|---|
| (A) H.C.F of the smallest composite number and the smallest prime number | (p) 6 |
| (B) H.C.F of 336 and 54 | (q) 5 |
| (C) H.C.F of 475 and 495 | (r) 2 |

Very Short Answer Questions

1. Write the condition to be satisfied by q so that a rational number $\frac{p}{q}$ has a terminating decimal expansion.
2. Write whether $\frac{2\sqrt{45}+3\sqrt{20}}{2\sqrt{5}}$ on simplification gives a rational or an irrational number.
3. The decimal expansion of the rational number $\frac{43}{2^4 \times 5^3}$ will terminate after how many places of decimals?
4. Has the rational number $\frac{441}{2^2 \times 5^7 \times 7^2}$ a terminating or a non-terminating decimal representation?
5. Given that H.C.F. of (306, 657) = 9, find the L.C.M. of (306, 657).
6. Find the H.C.F. of 12576 and 4052 by using the fundamental theorem of Arithmetic.
7. The length, breadth and height of a room are 8 m 25 cm, 6 m 75 cm and 4 m 50 cm, respectively. Determine the longest rod which can measure the three dimensions of the room exactly.
8. Write 98 as product of its prime factors.
9. Find the smallest positive rational number by which $\frac{1}{7}$ should be multiplied so that its decimal expansion terminates after 2 places of decimal.
10. Find a rational number between $\sqrt{2}$ and $\sqrt{3}$.

Short Answer Questions

1. Find the H.C.F. of 96 and 404 by the prime factorisation method. Hence, find their L.C.M.
2. Find the H.C.F. of 300, 540, 890 by applying Euclid's alogrithm.
3. Find the L.C.M. and H.C.F. of 336 and 54 by the prime factorization method.
4. Show that every positive even integer is of the form $2q$, and that every positive odd integer is of the form $2q + 1$, where q is some integer.
5. A, B and C starts cycling around a circular path in the same direction at same time. Circumference of the path is 1980 m. If the speed of A is 330 m/min, speed of B is 198 m/min and C is 220 m/min and they start from the same point, then after what time interval they will be together at the starting point?
6. Prove that $3+2\sqrt{5}$ is irrational.
7. Show that $5\sqrt{3}$ is an irrational number.
8. A certain number when divided by 899 leaves the remainder 63. Find the remainder when the same number is divided by 29.
9. What is the remainder when 4^{96} is divided by 6?
10. Write the H.C.F. of the smallest composite number and the smallest prime number.
11. Prove that $\sqrt{3}+\sqrt{2}$ is irrational.
12. Find the greatest number of six digits exactly divisible by 18, 24 and 36.
13. Find the H.C.F. of 180, 252 and 324 by Euclid's Division algorithm.

Long Answer Questions

1. Find the H.C.F of 81 and 237 and express it as a linear combination of 81 and 237.
2. Find the unit's digit in the product $7^{35} \times 3^{71} \times 11^{55}$.
3. A sweetseller has 420 kaju barfis and 130 badam barfis. She wants to stack them in such a way that each stack has the same number and they take up the least area of the tray. What is the maximum number of barfis that can be placed in each stack for this purpose?
4. Is it possible that H.C.F. and L.C.M of two numbers be 24 and 540 respectively. Justify your answer.
5. A fruit vendor has 990 apples and 945 oranges. He packs them into baskets. Each basket contains only one of the two fruits but in equal number. Find the number of fruits to be put in each basket in order to have minimum number of baskets.
6. Prove that $\sqrt{5}$ is an irrational number.

Reasoning Based Questions

1. If a and b are two positive integers such that the least prime factor of a is 3 and the least prime factor of b is 5. Then, find the least prime factor of $(a + b)$.
2. P is the L.C.M. of 2, 4, 6, 8, 10; Q is the LCM of 1, 3, 5, 7, 9 and L is the LCM of P and Q. Then, find the relation between L and P.
3. $P = 2(4)(6)$ ....(20) and $Q = 1(3)(5)$..............(19). What is the HCF of P and Q?
4. Given that the units digits of A^3 and A are the same, where A is a single digit natural number. How many possibilities can A assume?
5. If $X = 28 + (1 \times 2 \times 3 \times 4 \times$ ....$\times$ 28) and $Y = 17 + (1 \times 2 \times 3 \times$ ....$\times$ 17), then find out that $X + Y$ is a composite number or a prime number.

HOTS Questions

1. An army contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march ?
2. Prove that $\sqrt{p}+\sqrt{q}$ is irrational, where p and q are primes.
3. Prove that one of any three consecutive positive integers must be divisible by 3.

Exercise 2 Master NCERT (Textbook & Exemplar)

Text-book Questions

1. Use Euclid's division algorithm to find the H.C.F. of 135 and 225.

2. Show that any positive odd integer is of the from $6q + 1$, or $6q + 3$, or $6q + 5$, where q is some integer.

3. Using Euclid's division lemma, show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer m.

4. Find the L.C.M. and H.C.F. of the following integer by applying the prime factorisation method.
17 , 23 and 29

5. Given that H.C.F. (306, 657) = 9, find L.C.M (306, 657).

6. Check whether $6n$ can end with the digit 0 for any natural number n.

7. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.

8. Prove that the following are irrationals :
(i) $7\sqrt{5}$ (ii) $\dfrac{1}{\sqrt{2}}$

9. The following real numbers have decimal expansions as given below. In each case, decide whether they are rational or not. If they are rational and of the form $\dfrac{p}{q}$, what can you say about the prime factors of q?
(i) 0.120120012000120000...
(ii) 43.123456789

10. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$.

11. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the same point and at the same time and go in the same direction. After how many minutes will they meet again at the starting point?

Exemplar Questions

1. Write whether the square of any positive integer can be of the form $3m + 2$, where m is a natural number. Justify your answer.

2. Can two numbers have 18 as their H.C.F. and 380 as their L.C.M.? Give reason.

3. The numbers 525 and 3000 are both divisible only by 3, 5, 15, 25 and 75. What is the H.C.F. (525,3000)? Justify your answer.

4. Explain why $3 \times 5 \times 7 + 7$ is a composite number.

5. For some integer m, every even integer is of the form
(a) m (b) $m + 1$
(c) $2m$ (d) $2m + 1$

6. For some integer q, every odd integer is of the form
(a) q (b) $q + 1$
(c) $2q$ (d) $2q + 1$

7. The decimal expansion of the rational number $\dfrac{33}{2^2.5}$ will terminate after
(a) one decimal place
(b) two decimal places
(c) three decimal places
(d) more than 3 decimal places

8. Euclid's division lemma states that for two positive integers a and b, there exist unique integers q and r such that $a = bq + r$, where r must satisfy
(a) $1 < r < b$
(b) $0 < r \le b$
(c) $0 \le r < b$
(d) $0 < r < b$

9. On a morning walk, three persons step off together and their steps measure 40 cm, 42 cm and 45 cm, respectively. What is the minimum distance each should walk so that each can cover the same distance in complete steps?

Exercise 3 Foundation Builder

Single Option Correct

DIRECTIONS : *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct.*

1. When 2^{256} is divided by 17, then remainder would be
(a) 1 (b) 16
(c) 14 (d) None of these

2. The sum of three non-zero prime numbers is 100. One of them exceeds the other by 36. Then, the largest number is
(a) 73 (b) 91
(c) 67 (d) 57

3. The rational number of the form $\frac{p}{q}$, $q \neq 0$, p and q are positive integers , which represents $0.1\overline{34}$ i.e., (0.1343434....) is
(a) $\frac{134}{999}$ (b) $\frac{134}{990}$
(c) $\frac{133}{999}$ (d) $\frac{133}{990}$

4. The least number which is a perfect square and is divisible by each of 16, 20 and 24 is
(a) 240 (b) 1600
(c) 2400 (d) 3600

5. If n is an even natural number, then the largest natural number by which $n(n+1)(n+2)$ is divisible is
(a) 6 (b) 8
(c) 12 (d) 24

6. The least number which when divided by 15, leaves a remainder of 5, when divided by 25, leaves a remainder of 15 and when divided by 35, leaves a remainder of 25, is
(a) 515 (b) 525
(c) 1040 (d) 1050

7. The number $3^{13} - 3^{10}$ is divisible by
(a) 2 and 3 (b) 3 and 10
(c) 2, 3 and 10 (d) 2, 3 and 13

8. A number lies between 300 and 400. If the number is added to the number formed by reversing the digits, the sum is 888 and if the unit's digit and the ten's digit change places, the new number exceeds the original number by 9. Then, the number is
(a) 339 (b) 341
(c) 378 (d) 345

9. Which of the following will have a terminating decimal expansion?
(a) $\frac{77}{210}$ (b) $\frac{23}{30}$ (c) $\frac{125}{441}$ (d) $\frac{23}{8}$

10. I. The L.C.M. of x and 18 is 36.
II. The H.C.F. of x and 18 is 2.
What is the number x ?
(a) 1 (b) 2
(c) 3 (d) 4

11. If $a = 2^3 \times 3$, $b = 2 \times 3 \times 5$, $c = 3^n \times 5$ and L.C.M. $(a, b, c) = 2^3 \times 3^2 \times 5$, then $n =$
(a) 1 (b) 2
(c) 3 (d) 4

12. If p_1 and p_2 are two odd prime numbers such that $p_1 > p_2$, then $p_1^2 - p_2^2$ is
(a) an even number (b) an odd number
(c) an odd prime number (d) a prime number

13. When a natural number x is divided by 5, the remainder is 2. When a natural number y is divided by 5, the remainder is 4. The remainder is z when x + y is divided by 5. The value of $\frac{2z-5}{3}$ is **[NTSE]**
(a) –1 (b) 1
(c) –2 (d) 2

14. The largest non-negative integer k such that 24^k divides 13! is **[KVPY]**
(a) 2 (b) 3
(c) 4 (d) 5

15. On dividing a natural number by 13, the remainder is 3 and on dividing the same number by 21, the remainder is 11. If the number lies between 500 and 600, then the remainder on dividing the number by 19 is **[NTSE]**
(a) 4 (b) 6
(c) 9 (d) 13

16. Let $a_1, a_2, \ldots, a_{100}$ be non-zero real numbers such that $a_1 + a_2 + \ldots + a_{100} = 0$ **[KVPY]**
Then,
(a) $\sum_{i=1}^{100} a_i 2^{a_i} > 0$ and $\sum_{i=1}^{100} a_i 2^{-a_i} < 0$
(b) $\sum_{i=1}^{100} a_i 2^{a_i} \geq 0$ and $\sum_{i=1}^{100} a_i 2^{-a_i} \geq 0$
(c) $\sum_{i=1}^{100} a_i 2^{a_i} \leq 0$ and $\sum_{i=1}^{100} a_i 2^{-a_i} \leq 0$
(d) The sign of $\sum_{i=1}^{100} a_i 2^{a_i}$ or $\sum_{i=1}^{100} a_i 2^{-a_i}$ depends on the choice of a_i's

17. The value of $0.2\overline{35}$ is : **[JSTSE]**
(a) $\frac{233}{900}$ (b) $\frac{233}{990}$ (c) $\frac{235}{999}$ (d) $\frac{235}{990}$

18. The sum of all the possible remainders, which can be obtained when the cube of a natural number is divided by 9, is **[NTSE]**

(a) 5 (b) 6
(c) 8 (d) 9

19. Consider the following statements: For any integer n,
I. $n^2 + 3$ is never divisible by 17.
II. $n^2 + 4$ is never divisible by 17.
Then, **[KVPY]**

(a) both I and II are true
(b) both I and II are false
(c) I is false and II is true
(d) I is true and II is false

20. Given that $\frac{1}{7} = 0.\overline{142857}$, which is a repeating decimal having six different digits. If x is the sum of such first three positive integers n such that — $0.\overline{abcdef}$, where a, b, c, d, e and f are different digits, then the value of x is **[NTSE]**

(a) 20 (b) 21
(c) 41 (d) 42

21. The number of pairs (a, b) of positive real numbers satisfying $a^4 + b^4 < 1$ and $a^2 + b^2 > 1$ is **[KVPY]**

(a) 0 (b) 1
(c) 2 (d) More than 2

22. If $m = n^2 - n$, where n is an integer, then $m^2 - 2m$ is divisible by **[NTSE]**

(a) 20 (b) 24
(c) 30 (d) 16

23. Let N be the least positive integer such that whenever a non-zero digit c is written after the last digit of N, the resulting number is divisible by c. The sum of the digits of N is **[KVPY]**

(a) 9 (b) 18
(c) 27 (d) 36

24. The unit digit in the expression $55^{725} + 73^{5810} + 22^{853}$ is **[NTSE]**

(a) 0 (b) 4
(c) 5 (d) 6

More Than One Option Correct

DIRECTIONS : *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which one or more may be correct.*

1. Product of two co-prime numbers is 117. Their L.C.M. should be

(a) 1
(b) 117
(c) equal to their H.C.F.
(d) Lies between 115 to 120

2. Which of the following statement(s) is/are always false?

(a) The sum of two distinct irrational numbers is rational.
(b) The rationalising factor of a number is unique.
(c) Every irrational number is a surd.
(d) None of these

3. Which of the following statement(s) is/are correct?

(a) $\frac{7^3}{5^4}$ is a non-terminating repeating decimal.
(b) If $a = 2 + \sqrt{3}$ and $b = \sqrt{2} - \sqrt{3}$, then $a + b$ is irrational.
(c) If 19 divides a^3, then 19 divides a, where a is a positive integer.
(d) Product of L.C.M. and H.C.F. of 25 and 625 is 15625.

4. The product of unit digit in $(7^{95} - 3^{58})$ and $(7^{95} + 3^{58})$ is

(a) cube of 2
(b) lies between 6 and 10
(c) 6
(d) lies between 3 and 6

5. Which of the following statement(s) is/are correct?

(a) Every integer is a rational number.
(b) The sum of a rational number and an irrational number is an irrational number.
(c) Every real number is rational.
(d) Every point on a number line is associated with a real number.

6. Which of the following statement(s) is/are correct?

(a) There are infinitely many positive primes.
(b) Let 'a' be a positive integer and p be a prime number such that a^2 is divisible by p, then a is divisible by p.
(c) Every positive integer different from 1 can be expressed as a product of non-negative power of 2 and an odd number.
(d) If 'p' is a positive prime, then $\sqrt{p}$ is an irrational number.

Passage Based Questions

DIRECTIONS : *Study the given paragraph(s) and answer the following questions.*

Passage - I

If p is prime, then $\sqrt{p}$ is irrational and if a, b are two odd prime numbers, then $a^2 - b^2$ is composite.

1. $\sqrt{7}$ is

(a) a rational number
(b) an irrational number
(c) not a real number
(d) terminating decimal

2. $119^2 - 111^2$ is

(a) prime number
(b) composite
(c) an odd prime number
(d) an odd composite number

Passage - II

L.C.M. of several fractions $= \dfrac{\text{L.C.M. of their numerators}}{\text{H.C.F. of their denominators}}$

H.C.F. of several fractions $= \dfrac{\text{H.C.F. of their numerators}}{\text{L.C.M. of their denominators}}$

3. The L.C.M. of the fractions $\dfrac{5}{16}, \dfrac{15}{24}$ and $\dfrac{25}{8}$ is

(a) $\dfrac{5}{48}$ (b) $\dfrac{5}{8}$

(c) $\dfrac{75}{48}$ (d) $\dfrac{75}{8}$

4. The H.C.F of $\dfrac{2}{5}, \dfrac{6}{25}$ and $\dfrac{8}{35}$ is

(a) $\dfrac{2}{5}$ (b) $\dfrac{24}{5}$

(c) $\dfrac{2}{175}$ (d) $\dfrac{24}{175}$

5. The H.C.F. of the fractions $\dfrac{8}{21}, \dfrac{12}{35}$ and $\dfrac{32}{7}$ is

(a) $\dfrac{4}{105}$ (b) $\dfrac{192}{7}$

(c) $\dfrac{4}{7}$ (d) $\dfrac{5}{109}$

Assertion & Reason

DIRECTIONS : *Each of these questions contains an Assertion followed by Reason. Read them carefully and answer the question on the basis of following options. You have to select the one that best describes the two statements.*

(a) If both **Assertion** and **Reason** are **correct** and Reason is the **correct explanation** of Assertion.
(b) If both **Assertion** and **Reason** are correct, but Reason is **not the correct explanation** of Assertion.
(c) If **Assertion** is **correct** but **Reason** is **incorrect**.
(d) If **Assertion** is **incorrect** but **Reason** is **correct**.

1. **Assertion :** $\dfrac{13}{3125}$ is a terminating decimal fraction.
Reason : If $q = 2^n.5^m$ where n, m are non-negative integers, then $\dfrac{p}{q}$ is a terminating decimal fraction.

2. **Assertion :** Denominator of 34.12345. When expressed in the form $\dfrac{p}{q}, q \neq 0$, is of the form $2^m \times 5^n$, where m, n are non-negative integers.
Reason : 34.12345 is a terminating decimal fraction.

3. **Assertion :** The H.C.F. of two numbers is 16 and their product is 3072. Then, their L.C.M = 162.
Reason : If a, b are two positive integers, then H.C.F × L.C.M. = $a \times b$.

4. **Assertion :** 2 is a rational number.
Reason : The square roots of all positive integers are irrationals.

5. **Assertion :** If L.C.M. $\{p, q\} = 30$ and H.C.F $\{p, q\} = 5$, then $p.q = 150$.
Reason : L.C.M. of (a, b) × H.C.F of $(a, b) = a.b$.

6. **Assertion :** $n^2 - n$ is divisible by 2 for every positive integer.
Reason : $\sqrt{2}$ is not a rational number.

7. **Assertion :** $n^2 + n$ is divisible by 2 for every positive integer n.
Reason : If x and y are odd positive integers, from $x^2 + y^2$ is divisible by 4.

Multiple Matching Question

DIRECTIONS : *Following question has four statements (A, B, C and D) given in Column I and six statements (p, q, r, s, t and u) in Column-II. Any given statement in Column-I can have correct matching with one or more statement(s) given in Column-II.*

1.

Column-I	Column-II
(A) $\dfrac{551}{2^3 \times 5^6 \times 7^9}$	(p) a prime number
(B) Product of $\left(\sqrt{5}-\sqrt{3}\right)$ and $\left(\sqrt{5}+\sqrt{3}\right)$ is	(q) is an irrational number
(C) $\sqrt{5}-4$	(r) is a terminating decimal representation
(D) $\dfrac{422}{2^3 \times 5^4}$	(s) a rational number
	(t) is a non-terminating but repeating decimal representation
	(u) is non-terminating and non recurring decimal representation

Integer Type Questions

DIRECTIONS : *Answer the following questions. The answer to each of the question is a single digit integer, ranging from 0 to 9.*

1. Find the H.C.F. of 306 and 657.

2. Three numbers are in the ratio 1:2:3 and their H.C.F. is 12. Then, the square root of largest number is

Exercise 4 Foundation Builder +

DIRECTIONS (Qs. 1-7) : *This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct.*

1. H.C.F. of $(x^3 - 3x + 2)$ and $(x^2 - 4x + 3)$ is
(a) $(x - 1)$
(b) $(x - 2)^2$
(c) $(x - 1)(x + 2)$
(d) $(x - 1)(x - 3)$

2. In order that the number 1 y 3 y 6 be divisible by 11, the digit y should be
(a) 1 (b) 2
(c) 5 (d) 6

3. Number that has to be added to 345670 in order to make it divisible by 6 is
(a) 2 (b) 4
(c) 5 (d) 6

4. The greatest number of five digits exactly divisible 279 is
(a) 99603 (b) 99837
(c) 99882 (d) None

5. The greatest number which can divide 1854, 1866 and 2066 leaving the same remainder 2 in each case is
(a) 4 (b) 6
(c) 12 (d) None

6. If N is the sum of first 13,986 prime numbers, then N is always divisible by
(a) 6 (b) 4
(c) 8 (d) None of these

7. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to: **[NTSE]**
(a) 6 (b) 8
(c) 4 (d) 14

SOLUTIONS
(Brief Explanations of Selected Questions)

Exercise 1 **Master Boards**

Multiple Choice Questions

1. **(a)** Required number = H.C.F. {(70 – 5), (125 – 8)} = H.C.F. (65, 117) = 13.
2. **(b)** Required number = H.C.F.{(245 – 5), (1029 – 5)} = H.C.F. (240, 1024) = 16.
3. **(c)** H.C.F. of 20 and 15 = 5
   So, 5 students are in each group.
   $\therefore \quad n = \frac{20+15}{5} = \frac{35}{5} = 7$
   Hence, x = 4, y = 3 and n = 7
4. **(c)** 1
5. **(a)** H.C.F. (91, 126) = $\frac{91 \times 126}{\text{L.C.M.}(91, 126)} = \frac{91 \times 126}{182} = 13$
6. **(c)** $x = qy + r \Rightarrow 27 = 5 \times 5 + 2 \Rightarrow q = 5, r = 2$
7. **(d)** All the given statements are false.
8. **(b)** $196 = 2^2 \cdot 7^2$, sum of exponents = 2 + 2 = 4
9. **(c)** $0 \le r < b$

Fill in the Blanks

1. irrational
2. irrational
3. 4
4. irrational
5. irrational
6. algorithm
7. lemma
8. 9696
9. 6
10. $2^2 \times 3 \times 13$
11. terminating

True / False

1. True
2. True
3. True
4. True
5. True
6. False
7. True
8. True
9. True

Match the Following

1. (A) → (r) [∵ 12 = 3 × 4 ∵ it is a composite number]
   (B) → (s) [∵ g.c.d. between 2 and 7 = 1]
   (C) → (p) [∵ 2 is a prime number]
   (D) → (q) [∵ $\sqrt{2}$ is not a rational number]
2. (A) → (r); (B) → (p); (C) → (q)

Very Short Answer Questions

1. The prime factorization of q must be of the form $2^m \times 5^n$, where m, n are non-negative integers.
2. Rational number.
3. After 4 places of decimals.
4. Non-terminating
5. 22338
6. H.C.F. = $2^2 = 4$, L.C.M = (12576, 4052) = 1, 27, 39, 488
7. 75 cm.
8. $2 \times 7^2 = 98$
9. Since, $\frac{1}{7} \times \frac{7}{100} = \frac{1}{100} = 0.01$.
   The smallest rational number is $\frac{7}{100}$
10. Since, $\sqrt{2} = 1.414....$ and $\sqrt{3} = 1.732$ ...
    Hence, the rational number between $\sqrt{2}$ and $\sqrt{3}$ is 1.5 or $\frac{3}{2}$.

Short Answer Questions

1. The prime factorisation of 96 and 404 is:
   $96 = 2^5 \times 3$, $404 = 2^2 \times 101$
   Therefore, the H.C.F. of these two integers is $2^2 = 4$.
   ∵ H.C.F. $(a, b) \times$ L.C.M. $(a, b) = a \times b$
   ∴ L.C.M. (96, 404)
   $= \frac{96 \times 404}{\text{H.C.F.}(96, 404)} = \frac{96 \times 404}{4} = 9696$
2. ∵ 890 > 540, then by Euclid's Lemma
   $890 = 540 \times 1 + 350$
   $540 = 350 \times 1 + 190$
   $350 = 190 \times 1 + 160$
   $190 = 160 \times 1 + 30$
   $160 = 30 \times 5 + 10$
   $30 = 10 \times 3 + 0$
   Hence, H.C.F. of 890 and 540 is 10.
   Now, find the H.C.F. of 10 and third number 300.
   Since, 300 > 10, then by Euclid's Lemma
   $300 = 10 \times 30 + 0$
   Then, 10 is H.C.F. of 10 and 300
   Hence, required H.C.F. of 300, 540 and 890 is 10.
3. We have, $336 = 2^4 \times 3^1 \times 7^1$, $54 = 2^1 \times 3^3$
   ∴ H.C.F. of (336, 54) = $2^1 \times 3^1 = 6$
   L.C.M. of (336, 54) = $2^4 \times 3^3 \times 7 = 3024$
4. Let a be any positive integer and $b = 2$. Then, by Euclid's algorithm, $a = 2q + r$, for some integer $q \ge 0$, and $r = 0$ or $r = 1$, because $0 \le r < 2$. So, $a = 2q$ or $2q + 1$.
   If a is of the form $2q$, then a is an even integer. Also, a positive integer can be either even or odd. Therefore, any positive odd integer is of the form $2q + 1$.
5. Time taken by $A = \frac{\text{Distance}}{\text{Speed}} = \frac{1980}{330} = 6$ min
   Time taken by $B = \frac{1980}{198} = 10$ min
   Time taken by $C = \frac{1980}{220} = 9$ min

Hence, the required time after which A, B and C will be together at the starting point = L.C.M. (6, 10, 9) = 90 min

6. Let us assume on the contrary that $3+2\sqrt{5}$ is rational. Then, there exist co-prime positive integers a and b such that

$$3+2\sqrt{5}=\frac{a}{b} \Rightarrow 2\sqrt{5}=\frac{a}{b}-3 \Rightarrow \sqrt{5}=\frac{a-3b}{2b}$$

$\Rightarrow$ $\sqrt{5}$ is rational

$\left[\because a, b \text{ are integers } \therefore \frac{a-3b}{2b} \text{ is a rational}\right]$

This contradicts the fact that $\sqrt{5}$ is irrational. So, our supposition is incorrect.

Hence, $3+2\sqrt{5}$ is an irrational number.

7. If possible, let $5\sqrt{3}$ be a rational number.

So, $5\sqrt{3} = p/q$ where p and q are co-prime integers and $q \neq 0$

So, $\sqrt{3}=\frac{p}{5q}$

So, R.H.S is a rational number and hence $\sqrt{3}$ is also rational which is a contradiction.

So our supposition is wrong. Hence, $5\sqrt{3}$ is an irrational number.

8. Dividend = Divisor × Quotient + Remainder

Hence, remainder = 5 when same no. is divided by 29.

9. We know that the remainder for any power of 4 will be 4 only when divided by 6.

10. The smallest composite number $= 4 = 2 \times 2 = 2^2$

The smallest prime number $= 2 = 2^1 \therefore$ H.C.F. $= 2^1 = 2$

11. Let $\sqrt{3}+\sqrt{2} = r$ be a rational number.

$$\Rightarrow 3+2+2\sqrt{6}=r^2 \Rightarrow 2\sqrt{6}=r^2-5 \Rightarrow \sqrt{6}=\frac{r^2-5}{2}$$

As R.H.S is rational, $\sqrt{6}$ should be rational which is incorrect.

Hence, $\sqrt{3}$ $\sqrt{2}$ is an irrational number.

12. L.C.M. of 18, 24 and 36 is 72.

$$\begin{array}{r|l} 72 & 999999\,(13888 \\ & \underline{999936} \\ & \quad 63 \end{array}$$

$\therefore$ Required number = 9,99,936.

13. H.C.F. of 324 and 252 = 36

H.C.F. of 36 and 180 = 36

$\therefore$ H.C.F. of 180, 252 and 324 is 36.

Long Answer Questions

1.
$237 = 81 \times 2 + 75$ ... (i)
$81 = 75 \times 1 + 6$ ... (ii)
$75 = 6 \times 12 + 3$ ... (iii)
$6 = 3 \times 2 + 0$ ... (iv)

$\therefore$ H.C.F of 81 and 237 is 3.

To represent the H.C.F as a linear combination of the given two numbers, we start from the last but one step and successively eliminate the previous remainders as follows:

From (iii), we have

$3 = 75 - 6 \times 12$

$\Rightarrow 3 = 75 - (81 - 75 \times 1) \times 12$

[Substituting $6 = 81 - 75 \times 1$ obtained from (ii)]

$\Rightarrow 3 = 75 - 12 \times 81 + 12 \times 75$

$\Rightarrow 3 = 13 \times 75 - 12 \times 81$

[Substituting $75 = 237 - 81 \times 2$ obtained from (i)]

$\Rightarrow 3 = 13 \times (237 - 81 \times 2) - 12 \times 81$

$\Rightarrow 3 = 13 \times 237 - 26 \times 81 - 12 \times 81$

$\Rightarrow 3 = 13 \times 237 - 38 \times 81$

$\Rightarrow 3 = 237x + 81y$, where $x = 13$ and $y = -38$.

2. Unit digit in $(7^4) = 1$. Therefore, unit digit in $(7^4)^8$ i.e., 7^{32} will be 1. Hence, unit digit in

$$(7)^{35} = (7)^{32+3} = (7)^{32}(7)^3 = 1\times7\times7\times7 = 3$$

Again, unit digit in $(3)^4 = 1$

Therefore, unit digit in the expansion of

$$(3^4)^{17} = (3)^{68} = 1$$

$\Rightarrow$ Unit digit in the expansion of

$$(3^{71}) = (3)^{68+3} = (3)^{68}\cdot(3)^3 = 1\times3\times3\times3 = 7$$

and unit digit in the expansion of $(11^{35}) = 1$

Hence, unit digit in the expansion of

$7^{35} \times 3^{71} \times 11^{55} = 3 \times 7 \times 1 = 1$

3. Use Euclid's algorithm to find the H.C.F.

We have:

$420 = 130 \times 3 + 30$; $130 = 30 \times 4 + 10$; $30 = 10 \times 3 + 0$

So, the H.C.F. of 420 and 130 is 10.

Therefore, the sweet seller can make stacks of 10 for both kinds of barfi.

4. It is given that H.C.F. = 24, L.C.M. = 540

Now, $\frac{\text{L.C.M.}}{\text{H.C.F.}} = \frac{540}{24} = 22.5$ not an integer

Since, L.C.M. is always a multiple of H.C.F., hence, two numbers cannot have H.C.F. and L.C.M. as 24 and 540 respectively.

5. H.C.F. of 990 and 945 is 45.

The fruit vendor should put 45 fruits in each basket to have minimum number of baskets.

6. Let $\sqrt{5}$ is rational number

$\therefore \quad \sqrt{5} = \frac{p}{q}$,

where p and q are coprime integers and $q \neq 0$.

$$\sqrt{5}=\frac{p}{q} \Rightarrow p=\sqrt{5}q$$

Squaring both sides

$p^2 = 5q^2$ ...(i)

So, p^2 is divisible by 5

Then p is also divisible by 5

Let $p = 5$ m

Putting in (i), $(5m)^2 = 5q^2$; $25m^2 = 5q^2$; $q^2 = 5m^2$
So, q^2 is divisible by 5
Then q is also divisible by 5
Thus, p and q both divisible by 5 but p and q are coprime integers. By contradiction. $\sqrt{5}$ is irrational number.
Hence, proved.

Reasoning Based Questions

1. Clearly, 2 is neither a factor of a nor that of b.
$\therefore$ a and b are both odd. Hence, $(a + b)$ is even.
$\therefore$ least prime factor of $(a + b)$ is 2.

2. P is the L.C.M. of 2, 4, 6, 8, 10.
$\therefore P = 3(8)(5)$
Q is the L.C.M. of 1, 3, 5, 7, 9.
$\therefore Q = 5(7)(9)$
L is the L.C.M. of P and Q
$\therefore L = 3(8)(5)21 = 21P$

3. In P, the prime numbers that occur are 2, 3, 5, 7.
In Q, there are no 2's. So, the H.C.F. of P, Q has only 3's, 5's, 7's.
The 3's in P come from 6, 12, 18, i.e., 3^4 is the greatest power of 3 that is a factor of P, while for Q the 3's come from 3, 9, 15, i.e., 3^4 is also a factor of Q.
Similarly, 5^2 is the greatest power of 5 that for both P and Q and 7^1 is the greatest power of 7 for both P and Q.
$\therefore$ The H.C.F. of P and Q is $3^4(5^2)(7)$.

4. Since, A is a single digit natural number.
Then, A can be any of these 1, 2, 3, 4, 5, 6, 7, 8, 9.
Now, $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$,
$5^3 = 125$, $6^3 = 216$, $7^3 = 343$, $8^3 = 512$, $9^3 = 729$
Since, it is given that the unit digit of A^3 and A are the same. Then, the possible values of A are 1, 4, 5, 6 and 9.
Hence, required number of possibilities that A can assume is 5.

5. $X = 28 + (1 \times 2 \times 3 \times ... \times 16 \times 28)$
$\Rightarrow X = 28\ [1 + (1 \times 2 \times 3 \times ... 16)]$
$\therefore$ X is a composite number.
$Y = 17 + (1 \times 2 \times 3 \times 4 \times ... \times 17) = 17\ [1 + (1 \times 2 \times ... \times 16)]$
$\therefore$ Y is a composite number.
Now, $X + Y = [1 + (1 \times 2 \times 3 \times .... \times 16)]\ [28 + 17]$
$= [1 + (1 \times 2 \times 3 \times ... \times 16)]\ (45)$
Hence, $X + Y$ is a composite number.

HOTS Questions

1. The number n of the columns should be selected in such a way that the value of n be maximum and it must divide both the numbers, *i.e.,* 616 and 32.
So, n = H.C.F. (616, 32).
H.C.F. (616, 32) = 8
$\Rightarrow$ $n = 8$
Hence, the maximum number of the columns in which they can march = 8.

2. If possible, let $\sqrt{p} + \sqrt{q}$ be a rational number.
$\therefore$ $\sqrt{p} + \sqrt{q} = \frac{a}{b}$, where a and b are integers having no common factor other than 1 and $b \neq 0$.

$$\left(\sqrt{p} + \sqrt{q}\right)^2 = \frac{a^2}{b^2} \quad \text{[Squaring both sides]}$$

$$\Rightarrow \quad p + q + 2\sqrt{pq} = \frac{a^2}{b^2} \Rightarrow 2\sqrt{pq} = \frac{a^2}{b^2} - p - q$$

$$2\sqrt{pq} = \frac{a^2 - pb^2 - qb^2}{b^2} \Rightarrow \sqrt{pq} = \frac{a^2 - pb^2 - qb^2}{2b^2}$$

Now, $\sqrt{pq}$ is an irrational and $\frac{a^2 - pb^2 - qb^2}{2b^2}$ is a rational number.
This contradiction has arisen because of our incorrect assumption that $\sqrt{p} + \sqrt{q}$ is rational.
Hence, $\sqrt{p} + \sqrt{q}$ is an irrational number.

3. Let a be any positive integer. On dividing a by 3, let q be the quotient and r be the remainder.
By Euclid's Lemma, we have
$a = 3q + r$, where $0 \leq r < 2 \Rightarrow a = 3q + r$, where $r = 0, 1, 2$
(i) When $r = 0$, $a = 3q$, the three consecutive positive integers are
$n = 3q$, $n + 1 = 3q + 1$ and $n + 2 = 3q + 2$
Out of these, $3q$ is divisible by 3, $3q + 1$ is not divisible by 3 and $3q + 2$ is also not divisible by 3.
(ii) When $r = 1$, $a = 3q + 1$, the three consecutive positive integers are
$n = 3q + 1$, $n + 1 = 3q + 2$ and $n + 2 = 3q + 3$
Out of these, $3q + 1$ is not divisible by 3, $3q + 2$ is not divisible by 3 and $3q + 3 = 3(q + 1)$ is divisible by 3.
(iii) When $r = 2$, $a = 3q + 2$, the three consecutive positive integers are

$$n = 3q + 2, n + 1 = 3q + 3 = 3(q + 1)$$

and $\quad n + 2 = 3q + 4 = 3(q + 1) + 1$

Out of these, $3q + 2$ is not divisible by 3, $3q + 3 = 3\ (q + 1)$ is divisible by 3 and $3q + 4 = 3(q + 1) + 1$ is not divisible by 3.
Hence, one of three consecutive positive integers must be divisible by 3.

Exercise 2 Master NCERT (Textbook & Exemplar)

Text-book Questions

1. $225 = 135 \times 1 + 90$; $135 = 90 \times 1 + 45$; $90 = 45 \times 2 + 0$
H.C.F. (225, 135) = H.C.F. (135 , 90)
= H.C.F. (90, 45) = 45
Hence, the H.C.F. (225, 135) = 45

2. Let a be positive odd integer. By division algorithm,
$a = 6 \times q + r; 0 \leq r < 6 \Rightarrow a = 6q + 1$ or $6q + 1$
or $6q + 2$ or $6q + 3$ or $6q + 4$ or $6q + 5$.
We reject $6q + 0$, $6q + 2$ and $6q + 4$ because they are even integers.
Hence, the possible form of a is $6q + 1$
or $6q + 3$ or $6q + 5$.

3. Let n be a positive integer.
If $n = 3q \Rightarrow n^2 = (3q)^2 = 3 \times 3q^2 = 3m$, where $m = 3q^2$
If $n = 3q + 1 \Rightarrow n^2 = 3\,(3q^2 + 2q) + 1 = 3m + 1$,
where $m = 3q^2 + 2q$
If $n = 3q + 2 \Rightarrow n^2 = 3\,(3q^2 + 4q + 1) + 1 = 3m + 1$,
here $m = 3q^2 + 4q + 1$
Thus, n^2 is either in the form of $3m$ or $3m + 1$.

4. First we write the prime factorisation of each of the given numbers.
$17 = 17$, $23 = 23$ and $29 = 29$
$\therefore$ L.C.M. $= 17 \times 23 \times 29 = 11339$ and, H.C.F. $= 1$

5. We know that H.C.F. (a, b) × L.C.M. (a, b) = a × b

$$\Rightarrow \text{L.C.M.}(a, b) = \frac{a \times b}{\text{H.C.F.}(a, b)}$$

$$\therefore \text{L.C.M.}(306, 657) = \frac{306 \times 657}{\text{H.C.F.}(306, 657)}$$

$$= \frac{306 \times 657}{9} = 34 \times 657 = 22338$$

6. Let if possible 6^n ends with digit 0.
$\Rightarrow \; 6^n = 10 \times q \; \Rightarrow 2^n \times 3^n = 2 \times 5 \times q$
$\Rightarrow$ 5 is a prime factor of $2^n \times 3^n$
which is not possible because $2^n \times 3^n$ can have only 2 and 3 prime factors. Hence, not possible.

7. (i) $7 \times 11 \times 13 + 13 = 13 \times \{7 \times 11 + 1\}$
$= 13 \times 78$, which is a composite number.
(ii) $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5 = 5 \times \{7 \times 6 \times 4 \times 3 \times 2 \times 1 + 1\} = 5 \times 1009$, which is a composite number.

8. (i) Let us assume that $7\sqrt{5}$ be rational number. Then $7\sqrt{5}$ can be written in the form

$7\sqrt{5} = \dfrac{p}{q}$ where p, q $\in$ I and q $\neq$ 0

$$\Rightarrow \quad \sqrt{5} = \frac{p}{7q}$$

$\Rightarrow \quad \sqrt{5}$ is rational. It gives contradiction to our supposition that $7\sqrt{5}$ is a rational number.
Therefore, our supposition that $7\sqrt{5}$ is a rational number is wrong. Hence, $7\sqrt{5}$ is an irrational number.

(ii) $\dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

If possible let $\dfrac{\sqrt{2}}{2}$ be rational.

Then, $\dfrac{\sqrt{2}}{2} = \dfrac{p}{q}$, where p and q are integers

(q $\neq$ 0) and $\dfrac{2p}{q} = \sqrt{2}$

Since 2, p and q are integers, we get $\dfrac{2p}{q}$ is rational and so, $\sqrt{2}$ is rational.
But this contradicts the fact that $\sqrt{2}$ is irrational.

$\therefore$ Our assumption that $\dfrac{\sqrt{2}}{2}$ is rational, is wrong.

$\therefore \; \dfrac{\sqrt{2}}{2}$ is irrational

9. (i) Irrational.
(ii) Rational, $q = 10^9 = 2^9 \times 5^9$.

10. Let n be a positive integer.
Using Euclid Division Lemma, we can write 'n' as :
$n = 3q + r,\; 0 \le r < 3$
$\therefore$ n can be of the form: $3q$ or $3q + 1$ or $3q + 2$
If $\quad n = 3q$, $n^3 = 27q^3 = 9 \times 3q^3 = 9\,m$, where $m = 3q^3$
If $\quad n = 3q + 1 \Rightarrow n^3 = (3q + 1)^3$
$\Rightarrow n^3 = 9\{3q^3 + 3q^2 + q\} + 1$
$\because \; ((a + b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b)$
$= 9m + 1$, where $m = 3q^3 + 3q^2 + q$
If $\quad n = 3q + 2 \Rightarrow n^3 = (3q + 2)^3 = 27q^3 + 45q^2 + 36q$
$\Rightarrow \quad n^3 = 9(3q^3 + 6q^2 + 4q) + 8 = 9m + 8$, where
$m = 3q^3 + 6q^2 + 49$
Similarly, $n^3 = 9m + 8$
$\Rightarrow n^3$ is either of the form $9m$ or $9m + 1$ or $9m + 8$.

11. Let 'n' minutes be the required time.
So, first time, they will meet again in 'n' minutes. Second time, they will meet again in '$2n$' minutes & so on.
$\Rightarrow n$ is the minimum time required i.e; for 18 minutes & 12 minutes, 'n' will be the L.C.M. of 18 & 12, i.e; lowest number which is the multiple of 18 & 12 both.
Here, $18 = 2 \times 3 \times 3$ and $12 = 2 \times 2 \times 3$
$\therefore$ L.C.M. $(18, 12) = 2 \times 2 \times 3 \times 3 = 36$.
$\therefore$ Ravi and Sonia will meet again at the starting point after 36 minutes.

Exemplar Questions

1. No, because any positive integer can be written as $3q$, $3q + 1$, $3q + 2$, therefore, square will be $9q^2 = 3m$,
$9q^2 + 6q + 1 = 3\,(3q^2 + 2q) + 1 = 3m + 1$,
$9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1 = 3m + 1$.

2. No, because H.C.F. (18) does not divide L.C.M. (380).

3. The common factors of 525 and 3000 are 3, 5, 15, 25 and 75. The H.C.F. of 525 and 3000 is 75 as this is the greatest common factor.

4. We have, $3 \times 5 \times 7 + 7 = (3 \times 5 + 1) \times 7$
$= (15 + 1) \times 7 = 16 \times 7 = 112$
This is a composite number

5. **(c)** **6.** **(d)** **7.** **(b)** **8.** **(c)**

9. Step measures of three persons are 40 cm, 42 cm and 45 cm. The minimum distance each should walk so that each can cover the same distance in complete steps is the L.C.M. of 40 cm, 42 cm and 45 cm.
Prime factorisation of 40, 42 and 45 gives
$40 = 2^3 \times 5$, $42 = 2 \times 3 \times 7$, $45 = 3^2 \times 5$
L.C.M. (40, 42, 45) = Product of the greatest power of each prime factor involved in the numbers
$= 2^3 \times 3^2 \times 5 \times 7 = 8 \times 9 \times 35 = 72 \times 35 = 2520$ cm.

Exercise 3 Foundation Builder

Single Option Correct

1. **(a)** When 2^{256} is divided by 17 then, $\frac{2^{256}}{2^4+1}=\frac{(2^4)^{64}}{(2^4+1)}$

By remainder theorem when $f(x)$ is divided by $x+a$ the remainder $= f(-a)$

Here, $f(a)=(2^4)^{64}$ and $x=2^4$ and $a=1$

$\therefore$ Remainder $= f(-1)=(-1)^{64}=1$

2. **(c)** Since, the sum of all the three prime numbrs is 100. Then, there are two cases

Case 1: All the three numbers should be even because 100 is an even number. But this case is not possible as there is only one even prime.

Case 2 : One prime is even and other two primes are odd. Since, 2 is only even prime, so it must be one of three primes.

Let p and $p+36$ be the other two primes.

Then, according to question

$2+p+(p+36)=100$; $2p+38=100$

$2p=100-38=62$; $p=\frac{62}{2}=31$

So, all the three primes are 2, 31 and 67.

Hence, largest prime number is 67.

3. **(d)** $0.1\overline{34}=\frac{134-1}{990}=\frac{133}{990}$

4. **(d)** The L.C.M. of 16, 20 and 24 is 240. The least multiple of 240 that is a perfect square is 3600 and also we can easily eliminate choices (a) and (c) since they are not perfect number. Hence, the required least number which is also a perfect square is 3600 which is divisible by each of 16, 20 and 24.

5. **(d)** Out of n and $n+2$, one is divisible by 2 and the other by 4, hence $n\,(n+2)$ is divisible by 8. Also n, $n+1$, $n+2$ are three consecutive numbers, hence one of them is divisible by 3. Hence, $n\,(n+1)\,(n+2)$ must be divisible by 24. This will be true for any even number n.

6. **(a)** The number divisible by 15, 25 and 35 = L.C.M. (15, 25, 35) = 525

Since, the number is short by 10 for complete division by 15, 25 and 35.

Hence, the required least number = 525 – 10 = 515.

7. **(d)** $3^{13}-3^{10}=3^{10}(3^3-1)=3^{10}(26)=2\times 13\times 3^{10}$

Hence, $3^{13}-3^{10}$ is divisible by 2, 3 and 13

8. **(d)** Sum is 888 $\Rightarrow$ unit's digit should add up to 8. This is possible only for option (d) as "3" + "5" = "8".

9. **(d)**

10. **(d)** L.C.M × H.C.F = First number × second number

Hence, required number $=\frac{36\times 2}{18}=4$.

11. **(b)** Value of $n=2$

12. **(a)** Since, p_1 and p_2 are odd primes and sum of two odd number is an even number.

So, p_1+p_2 is an even number.

Since, multiple of even number is always even.

Therefore, $(p_1+p_2)\,(p_1-p_2)$ is even

Hence, $p_1^2-p_2^2=(p_1+p_2)(p_1-p_2)$ is an even number.

13. **(a)** Since, x is divided by 5, the remainder is 2 therefore

x = 5m + 2

similarly, y = 5n + 4

consider x + y = 5(m + n) + 6

= 5(m + n) + 5 + 1 = 5(m + n + 1) + 1

But given that when x + y is divided by 5, the remainder is z

$\therefore$ z = 1

Now, $\frac{2z-5}{3}=\frac{2\,(1)-5}{3}=-1$

14. **(b)** We know that

$13! = 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10\times 11\times 12\times 13$

$=2^{10}\times 3^5\times 5^2\times 7\times 11\times 13 \Rightarrow 24^k=(2^3\times 3)^k$

where k is largest non-negative integer

When 13! is an divided by 24^k, we get

$\frac{2^{10}\times 3^5\times 5^2\times 7\times 11\times 13}{2^{3k}.3^k}$

$=2^{10-3k}\,.\,3^{5-k}\,.\,5^2\times 7\times 11\times 13$

$\therefore$ $10-3k$ is integer.

Then, maximum value of $k=3$

15. **(a)** **Given:** The natural number, when divided by 13 leaves remainder 3

The natural number, when divided by 21 leaves remainder 11

So, 13 – 3 = 21 – 11 = 10 = k

Now, LCM (13, 21) = 273

But the number lies between 500 and 600

$\therefore$ 2 LCM (13, 21) – k = 546 – 10 = 536

536 = 19 × 8 + 4 $\therefore$ remainder = 4

16. **(a)** Let $a_1, a_2, a_3, \ldots, a_{100}$ be non-zero real number and $a_1+a_2+a_3+\ldots+a_{100}=0$

$a_i\cdot 2^{a_i}>a_i$ and $a_i\cdot 2^{-a_i}<a_i$

$\therefore \sum_{i=1}^{100}a_1\cdot 2^{a_i}>\sum_{i=1}^{100}a_i$ and $\sum_{i=1}^{100}a_1\cdot 2^{-a_i}<\sum_{i=1}^{100}a_i$

$\Rightarrow \sum_{i=1}^{100}a_1\cdot 2^{a_i}>0$ and $\sum_{i=1}^{100}a_1\cdot 2^{-a_i}<0$

Hence, option (a) is correct.

17. **(c)** Let $x=0.\overline{235}$ ...(i)

$1000x=235.\overline{235}$ ...(ii)

Subtract (i) from (ii), $999x=235\Rightarrow x=\frac{235}{999}$

18. **(d)** Let any no. P be divided by 3 possible result is 3q; 3q + 1 and 3q + 2 when p = 3q than $P^3=27q^3=9\,(3q^3)$ no remainder when P = 3q +1;

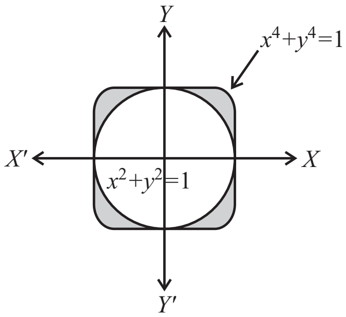
$P^3 = (3q + 1)^3 = (27q^3) + 9q\,(3q + 1) + 1$
$= 9(3q^3 + 3q^2 + q) + 1$ remainder = 1, When $P = (3q + 2)$
$P^3 = (3q + 2)^3 = 27q^3 + 18q\,(3q + 2) + 8$
$= 9(3q^3 + 2q\,(3q + 2)] + 8$ remainder = 8
Sum of remainder = 1 + 8 = 9

19. **(d)** Let us consider that $n^2 + 3$ is divisible by 17
$\therefore\ n^2 + 3 = 17K \quad [K \in N]$
$\Rightarrow\ n^2 = 17K - 3 \ \Rightarrow\ n^2 = 3\,(17m - 1) \quad [\because K = 3m]$
$3(17m - 1)$ is a perfect square, which is not possible.
$\therefore\ n^2 + 3$ is never divisible by 17.
In, $n^2 + 4$, put $n = 9$
So, $(9)^2 + 4 = 81 + 4 = 85$ which is divisible by 17.
$\therefore$ I is true and II is false.

20. **(c)** $\frac{1}{7} = 0.\overline{142857}$

The second positive integer whose reciprocal have six different repeating decimals is
$\frac{1}{13} = 0.\overline{076923}$
And the third positive integer whose reciprocal have six different repeating decimals is
$\frac{1}{21} = 0.\overline{047619}$
Therefore, the values of x are 7, 13, 21
Hence, the required sum is = 7 + 13 + 21 = 41

21. **(d)** We have, $a^4 + b^4 < 1$ and $a^2 + b^2 > 1$
The graphs of $x^2 + y^2 = 1$ and $x^4 + y^4 = 1$ are given below:



It is clear from graph, there are many positive real numbers (a, b) satisfying $a^4 + b^4 < 1$ and $a^2 + b^2 > 1$

22. **(b)** $\because \quad m = n^2 - n = n(n - 1)$
Now, $m^2 - 2m = m(m - 2)$
$= n(n - 1)(n^2 - n - 2) = n(n - 1)(n - 2)(n + 1)$
Since we know that product of any four consecutive integers is always divisible by 24.
$\therefore\ m^2 - 2m$ is divisible by 24.

23. **(a)** Sum of digits of N should be divisible by 9. In this case, the resulting number will always be divisible if c = 1, 2, 3, 5, 6 and 9.
To make the resulting number divisible by 4, 7 and 8 as well N should be a multiple of LCM of (4, 7, 8) i.e. 56.
Smallest positive integer which is multiple of 56 and whose sum of digits is a multiple of 9 is 504.
$\therefore$ N will be 504. $\quad \therefore$ Sum of digits = 9.

24. **(d)** For given numbers,
$(55)^{725}$, unit digit = 5; $(73)^{5810}$, unit digit = 9
$(22)^{853}$, unit digit = 2
Unit digit in the expression
$55^{725} + 735^{810} + 22^{853}$ is $\underline{6}$

More Than One Option Correct

1. **(b, d)** Since, H.C.F. of co-prime number is 1.
$\therefore$ Product of two co-prime numbers is equal to their L.C.M. So, L.C.M. = 117

2. **(a, b, c)** **3.** **(b, c, d)**

4. **(a, b)** Unit digit in (7^{95}) = Unit digit in $[(7^4)^{23} \times 7^3]$
= Unit digit in 7^3 (as unit digit in $7^4 = 1$)
= Unit digit in 343
Unit digit in 3^{58} = Unit digit in $(3^4)^4 \times 3^2$
[as unit digit $3^4 = 1$]
= Unit digit is 9
So, unit digit in $(7^{95} - 3^{58})$
= Unit digit in (343 – 9) = Unit digit in 334 = 4
Unit digit in $(7^{95} + 3^{58})$ = Unit digit in (343 + 9)
= Unit digit in 352 = 2
So, the product is $4 \times 2 = 8$

5. **(a, b, d)**

6. **(a, b, c, d)** All the statements are correct.

Passage Based Questions

1. **(b)** **2.** **(b)**

3. **(d)** L.C.M.of $\frac{5}{16}, \frac{15}{24}$ and $\frac{25}{8} = \frac{\text{L.C.M. of numerators}}{\text{H.C.F. of denominators}}$

L.C.M. of 5, 15 and 25 is 75.
H.C.F. of 16, 24 and 8 is 8.

$\therefore$ The L.C.M. of the given fractions = $\frac{75}{8}$

4. **(c)** H.C.F. of the fractions $= \frac{\text{H.C.F. of numerators}}{\text{L.C.M. of denominators}}$

H.C.F. of 2, 6 and 8 is 2.
L.C.M. of 5, 25 and 35 is 175.

Thus, the H.C.F. of the given fractions $= \frac{2}{175}$.

5. **(a)** H.C.F. of given fraction is

$= \frac{\text{H.C.F. of 8, 12, 32}}{\text{L.C.M. of 21, 35, 7}} = \frac{4}{105}$

Assertion & Reason

1. **(a)** Reason is correct.
Since, the factors of the denominator 3125 is of the form $2^0 \times 5^5$.

$\therefore \ \frac{13}{3125}$ is a terminating decimal

Since, assertion follows from reason.

2. **(a)** Reason is clearly true.

Again, $34.12345 = \frac{3412345}{100000} = \frac{682469}{20000} = \frac{682469}{2^5 \times 5^4}$

Its denominator is of the form $2^m \times 5^n$, where $m = 5$, $n = 4$ are non-negative integers

$\therefore$ Assertion is true. Since, reason gives assertion

$\therefore$ (a) holds.

3. **(d)** Here, reason is true [standard result]

Assertion is false. $\because \frac{3072}{16} = 192 \neq 162$

4. **(c)** Here, reason is not true.

$\because \sqrt{4} = \pm 2$, which is not an irrational number.

$\therefore$ Reason does not hold. Clearly, assertion is true.

5. **(a)** **6.** **(b)** **7.** **(a)**

Multiple Matching Questions

1. **(A) → (t, s); (B) → (p, s); (C) → (q, u); (D) → (r, s)**

Integer Type Questions

1. **Ans : 9**

$306 = 2 \times 3^2 \times 17$, $657 = 3^2 \times 73$

H.C.F. $(306, 657) = 3^2 = 9$

2. **Ans : 6**

Let the required numbers be y, $2y$ and $3y$.

H.C.F. $(y, 2y, 3y) = y \Rightarrow y = 12$

$\therefore$ The numbers are 12, 24 and 36.

$\therefore$ The square root of largest number $= \sqrt{36} = 6$.

Exercise 4 Foundation Builder +

1. **(a)** $x^3 - 3x + 2 = (x-1)(x^2 + x - 2)$

$= (x-1)(x-1)(x+2) = (x-1)^2(x+2)$

$x^2 - 4x + 3 = (x-1)(x-3)$

Hence, required H.C.F.

$[(x^3 - 3x + 2), (x^2 - 4x + 3)] = (x-1)$

2. **(c)** Sum of digits at even places $= y + y = 2y$

Sum of digits at odd places $= 1 + 3 + 6 = 10$

Now, by the divisibility test of 11.

Sum of digits at odd palces – sum of digits at even places $= 0$ or multiple of 11.

$\Rightarrow 10 - 2y = 0$ or multiple of 11 when, $10 - 2y = 0 \Rightarrow y = 5$.

3. **(a)** On dividing the given number 345670 by 6, we get 4 as the remainder.

Hence, required number that must be added to the given number is 2.

4. **(c)**

5. **(a)** The required greatest number

$=$ H.C.F. $[1854 + 2, 1866 + 2, 2066 + 2]$

$=$ H.C.F. $[1856, 1868, 2068] = 4$

6. **(d)** N will be an odd number because N is sum of one even number (2) and 13985 odd numbers.

Hence, N will not be divisible by even number.

7. **(b)** $2^{403} = 2^{400} . 2^3 = 2^{4 \times 100} . 2^3 = (2^4)^{100} . 8$

$= 8(2^4)^{100} = 8(16)^{100} = 8\,(1 + 15)^{100} = 8 + 15\,\mu$

When 2^{403} is divided by 15, then remainder is 8.

Hence, fractional part of the number is $\frac{8}{15}$

Therefore value of k is 8

2. Polynomials

Build A Strong Foundation

CONCEPT MAP

POLYNOMIALS

① Polynomial

An algebraic expression $f(x)$ of the form

$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$,

where $a_0, a_1, ...a_n$ are real numbers and all indices of variable x are non-negative integers is called polynomial in variable x.

(i) The highest indices of x is called degree of the polynomial.

(ii) $a_0, a_1x, ...a_nx^n$ are terms of the polynomial.

(iii) $a_0, a_1, ...a_n$ are coefficients of the polynomial.

② Types of Polynomial

On the basis of number of terms in the polynomial	On the basis of degree of the polynomial
(i) Monomial: If polynomial has one term. **(ii) Binomial:** If polynomial has two terms. **(iii) Trinomial:** If polynomial has three terms.	**(i) Linear:** If polynomial has degree one. **(ii) Quadratic:** If polynomial has degree two. **(iii) Cubic:** If polynomial has degree three. **(iv) Biquadratic:** If polynomial has degree four.

③ Value of a Polynomial

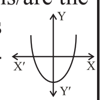
The value of a polynomial $f(x)$ at $x = \alpha$ is obtained by substituting $x = \alpha$ in the given polynomial and is denoted by $f(\alpha)$

④ Zero(es)/Root(s) of Polynomial

$x = r$ is a zero of a polynomial $p(x)$ if $p(r) = 0$

⑤ Geometrical Meaning of Zeroes of the Polynomial

Zero(es) of a polynomial is/are the x-coordinate of the points where graph $y = f(x)$ intersects the x-axis.



(i) Linear polynomial : Graph of linear polynomial is a straight line and has exactly one zero.

(ii) Quadratic polynomial : Graph of quadratic polynomial is always a parabola and this polynomial can have atmost two zeroes.

(iii) Cubic polynomial: Cubic polynomial can have atmost three zeroes.

⑥ Relationship between Zeroes and Coefficient of a Polynomial

(i) Zero of a linear polynomial ax + b is $x = -\frac{b}{a}$

(ii) If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, then

$\alpha + \beta = -\frac{b}{a}$

i.e. sum of zeroes $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

and $\alpha\beta = \frac{c}{a}$

i.e. product of zeroes $= \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

(iii) If α, β and γ are zeroes of the cubic polynomial, then $\alpha + \beta + \gamma = -\frac{b}{a}$

i.e. sum of zeroes $= -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

i.e. sum of product of every two root

$= \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$, $\alpha\beta\gamma = -\frac{d}{a}$

i.e. product of roots $= -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$

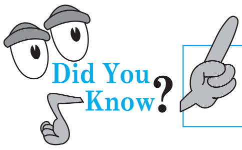
⑦ Division Algorithm

If $p(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that $p(x) = g(x) \times q(x) + r(x)$ where either $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$

STANDARD FORM OF A POLYNOMIAL

Here, $a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_nx^n$ or $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots\ldots\ldots + a_2x^2 + a_1x + a_0$ are standard form of the same polynomial in variable x, in which power of the variable are either increasing or decreasing.

- A symbol like x in the above polynomial, which takes various numerical values is known as a variable.
- In a polynomial, powers of the variable are non-negative integer.
- Terms having same variable with same indices are called like terms.
- Here, $a_0, a_1, a_2, \ldots\ldots\ldots\ldots, a_n$ are real numbers. If $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are all integers then the above polynomial is said to be a polynomial over integers.
- If $a_n \neq 0$, then degree of the above polynomial is n.



If we put any polynomial equal to zero then we get polynomial equation i.e. if we put $f(x) = 0$, then we get the polynomial equation as $a_nx^n + a_{n-1}x^{n-1} + \ldots . a_1x + a_0 = 0$

Examples :

(i) $6x^7 - 5x^4 + 2x + 3$ is a polynomial of degree 7.

(ii) $2 + 5x^{3/2} + 7x^2$ is an expression but not a polynomial , since it contains a term in which power of x is 3/2, which is not a non-negative integer.

(iii) In $-9y^2$, -9 is the coefficient of y; y is the variable and 2 is the power or index of y.

(iv) $5a^2, -7a^2$ and $\sqrt{2}a^2$ are like terms.

DIFFERENT TYPES OF POLYNOMIALS

There are some important types of polynomial based on the numebr of terms in it. These are listed below:

(i) **Monomial :** A polynomial having one term is called a monomial. **Example :** $3, 5x, 6a$ etc.

(ii) **Binomial :** A polynomial having two terms is called a binomial. **Example :** $2x^2 + 3$, $6y - b$, $3 + 7z$ etc.

(iii) **Trinomial :** A polynomial having three terms is called a trinomial. **Example :** $30x + by + cz$, $2a - 3x^2 + y^3$ etc.

There are some important types of polynomials based on degrees. These are listed below :

(i) **Linear polynomials :** A polynomial of degree one is called a linear polynomial. The general form of the linear polynomial is $ax + b$, where a and b are any real constant and $a \neq 0$.
Example : $3 + 5x$ is a linear polynomial.

(ii) **Quadratic polynomials :** A polynomial of degree two is called a quadratic polynomial. The general form of the quadratic polynomial is $ax^2 + bx + c$, where $a \neq 0$.
Example : $2y^2 + 3y - 1$ is a quadratic polynomial.

(iii) **Cubic polynomials :** A polynomial of degree three is called a cubic polynomial. The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where $a \neq 0$.
Example : $6x^3 - 5x^2 + 2x + 1$ is a cubic polynomial.

(iv) **Biquadratic polynomials :** A polynomial of degree four is called biquadratic polynomial. The general form of biquadratic polynomial is $ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$. **Example :** $4x^4 - 6x^3 + 3x^2$

ILLUSTRATION : 1

Which of the following expressions are polynomials ?

(a) $5x^2 - 3x + 9$ **(b)** $\frac{4}{3}x^7 - 5x^4 + 3x^2 - 1$

SOLUTION :

Both are polynomials.

ILLUSTRATION : 2

Write down the degree of the following polynomials.

(a) $3x + 5$ **(b)** $3t^2 - 5t + 9t^4$ **(c)** $2 - y^2 - y^3 + 2y^8$

SOLUTION :

(a) The highest power term is $3x$ and exponent of x is 1. $\therefore$ degree of polynomial $3x + 5$ is 1

(b) The highest power term is $9t^4$ and exponent of t is 4. So, degree of the given polynomial is 4.

(c) The highest power term is $2y^8$ and exponent of y is 8.

VALUE OF A POLYNOMIAL

A polynomial of x can be represented by $f(x)$, $g(x)$, $p(x)$, $q(x)$, etc. The value of a polynomial $f(x)$ at $x = \alpha$ is obtained by substituting $x = \alpha$ in the given polynomial and is denoted by $f(\alpha)$.

Consider the polynomial : $p(x) = 5x^3 - 2x^2 + 3x - 2$

If we replace x by 1 everywhere in $p(x)$, we get

$p(1) = 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2 = 5 - 2 + 3 - 2 = 4$

So, we say that the value of $p(x)$ at $x = 1$ is 4.

ZERO(ES) / ROOT(S) OF POLYNOMIALS

Let $x = r$ be a root or zero of a polynomial $P(x)$, if $P(r) = 0$.

In other words, $x = r$ is a root or zero of a polynomial if it is a solution to the equation $P(x) = 0$

The process of finding the zeroes of $P(x)$ is actually solving the equation $P(x) = 0$

Let's first find the zeroes of $P(x) = x^2 + 2x - 15$. To find the zeroes, we simply solve the following equation.

$x^2 + 2x - 15 = 0 \Rightarrow (x + 5)(x - 3) = 0 \Rightarrow x = -5, 3$

So, this second degree polynomial has two zeroes or roots –5 and 3.

ILLUSTRATION : 3

If $x = 2$ and $x = 0$ are roots of the polynomial $f(x) = 2x^3 - 5x^2 + ax + b$. Find the values of a and b.

SOLUTION :

Since, 2 is a root, therefore

$f(2) = 2(2)^3 - 5(2)^2 + a(2) + b = 0 \quad \Rightarrow \quad 16 - 20 + 2a + b = 0 \Rightarrow 2a + b = 4$

Since 0 is also a root, therefore

$f(0) = 2(0)^3 - 5(0)^2 + a(0) + b = 0 \quad \Rightarrow \quad b = 0 \Rightarrow 2a = 4 \Rightarrow a = 2, b = 0$

Time to Check Your Knowledge

☛ **If the zeroes of $p(x) = x^3 + 2x^2 + a$, are also the zeroes of polynomial $q(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x + b$; then find the values of a and b. Which zeroes of $q(x)$ are not the zeroes of $p(x)$?**

SOLUTION

If zeroes of $p(x)$ are also the zeroes of $q(x)$, then $q(x)$ is divisible by $p(x)$. Let us now divide $q(x)$ by $p(x)$ to obtain the remainder.

$$x^3 + 2x^2 + a \,\big)\, x^5 - x^4 - 4x^3 + 3x^2 + 3x + b \,\big(\, x^2 - 3x + 2$$

$$x^5 + 2x^4 + ax^2$$
$$- \quad - \quad -$$

$$-3x^4 - 4x^3 + (3 - a)x^2 + 3x + b$$
$$-3x^4 - 6x^3 - 3ax$$
$$+ \quad + \quad +$$

$$2x^3 + (3 - a)x^2 + 3x(1 + a) + b$$
$$2x^3 + 4x^2 \qquad + 2a$$
$$- \quad - \qquad -$$

$$(-1 - a)x^2 + 3x(1 + a) + b - 2a$$

If $q(x)$ is divisible by $p(x)$, then remainder must be 0. Now, remainder = 0

$\Rightarrow (-1-a)x^2 + 3x(1+a) + b - 2a = 0$ for all x.

$\Rightarrow (-1-a) = 0,\ 3(1+a) = 0$ and $b - 2a = 0 \Rightarrow a = -1$ and b = –2

Substituting a = –1 in $p(x)$, we obtain $p(x) = x^3 + 2x^2 + 1$

Now, $x^2 - 3x + 2 = (x-1)(x-2)$. So, zeroes of $x^2 - 3x + 2$ are 1 and 2.

$q(1) = 2 \neq 0$ and $q(2) = 15 \neq 0$

So, zeroes of the $x^2 - 3x + 2$ are not the zeroes of $p(x)$.

Hence, 1 and 2 are zeroes of $q(x)$ which are not the zeroes of $p(x)$.

GEOMETRICAL MEANING OF THE ZERO(ES) OF A POLYNOMIAL

Zero(es) of a polynomial is/are the x-coordinate of the point(s) where the graph of $y = f(x)$ intersects the x-axis.

GRAPHS OF POLYNOMIALS

Graph of a polynomial $f(x)$ is a smooth free hand curve passing through points (x_1, y_1), (x_2, y_2), (x_3, y_3), ... etc., where y_1, y_2, y_3, ... are the values of the polynomial $f(x)$ at x_1, x_2, x_3, ... respectively. In order to draw the graph of a polynomial $f(x)$, we may follow the following algorithm.

Algorithm to draw the graph of the polynomials :

Step 1 : Find the values $y_1, y_2, ...y_n$, that is value of polynomial $f(x)$ at $x_1, x_2, ...x_n$ and prepare a table that gives values of y or $f(x)$ for various values of x.

x	x_1	x_2	.....	x_n	x_{n+1}	.....
$y = f(x)$	$y_1 = f(x_1)$	$y_2 = f(x_2)$	.....	$y_n = f(x_n)$	$y_{n+1} = f(x_{n+1})$	.....

Step 2 : Plot these points (x_1, y_1), (x_2, y_2), (x_3, y_3), ...(x_n, y_n),......... on rectangular coordinate system. In plotting these points you may use different scales on the x and y-axis.

Step 3 : Draw a free hand smooth curve passing through the points plotted in step 2 to get the graph of the polynomial $f(x)$.

GRAPH OF A LINEAR POLYNOMIAL

Consider a linear polynomial $f(x) = ax + b$, $a \neq 0$. In class IX we have studied that the graph of $y = ax + b$ is a straight line that has exactly one zero. That is why $f(x) = ax + b$ is called a linear polynomial. Since two points determine a straight line, so only two points needed to plot or draw the line $y = ax + b$. The line represented by $y = ax + b$ crosses the x-axis at exactly one point, namely $\left(-\frac{b}{a}, 0\right)$.

ILLUSTRATION : 4

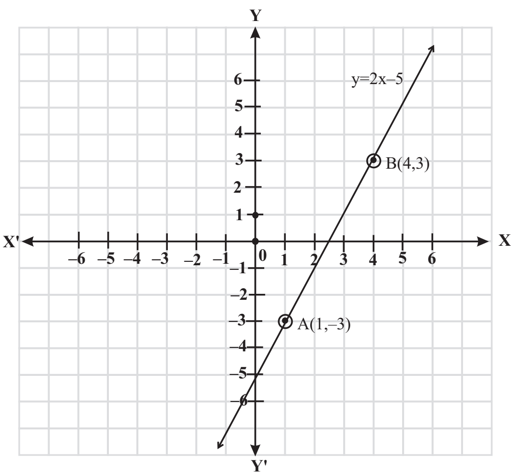
Draw the graph of the polynomial $f(x) = 2x - 5$. Also, find its zero.

SOLUTION :

Let $y = 2x - 5$

The following table list the values of y corresponding to different values of x.

x	1	4
y	–3	3



The points A (1, –3) and B (4, 3) are plotted on the graph paper on a suitable scale. A line is drawn passing through these points to obtain the graphs of the given polynomial.

The line intersects the X-axis at $x = \frac{5}{2}$. Hence, zero or root of the given polynomial is $\frac{5}{2}$.

GRAPH OF A QUADRATIC POLYNOMIAL

Let a, b, c be real numbers and $a \neq 0$. Then $f(x) = ax^2 + bx + c$ is known as quadratic polynomial in x and it has either no zero or two zeroes.

Example :

1. There are no zeroes of quadratic polynomial $f(x) = x^2 + 1$ and thus the graph of the polynomial doesn't intersect x-axis.
2. There are two zeroes of quadratic polynomial $p(x) = x^2 - 4x + 4$ i.e., 2, 2; which means both zeroes are same and thus the graph of this polynomial intersects x-axis only once at 2.
3. There are two zeroes of quadratic polynomial $g(x) = x^2 - 9$ that is +3, –3, thus the graph of this polynomial intersect x-axis at two places, 3 & – 3

Note : (i) The two zeroes of quadratic polynomial can be either both rational or both irrational.

(ii) Graph of a quadratic polynomial is always a parabola.

The graph shown in the illustration 5 and 6 are the parabola.

Some parabola are upward open as in illustration 5 and others are downward open as in illustration 6.

If the value of a in the quadratic polynomial $ax^2 + bx + c$ is positive, then the parabola is upward open. If the value of a in the quadratic polynomial $ax^2 + bx + c$ is negative, then the parabola is downward open.

ILLUSTRATION : 5

Draw the graph of the polynomial $f(x) = x^2 - 2x - 8$. Also find its zeroes.

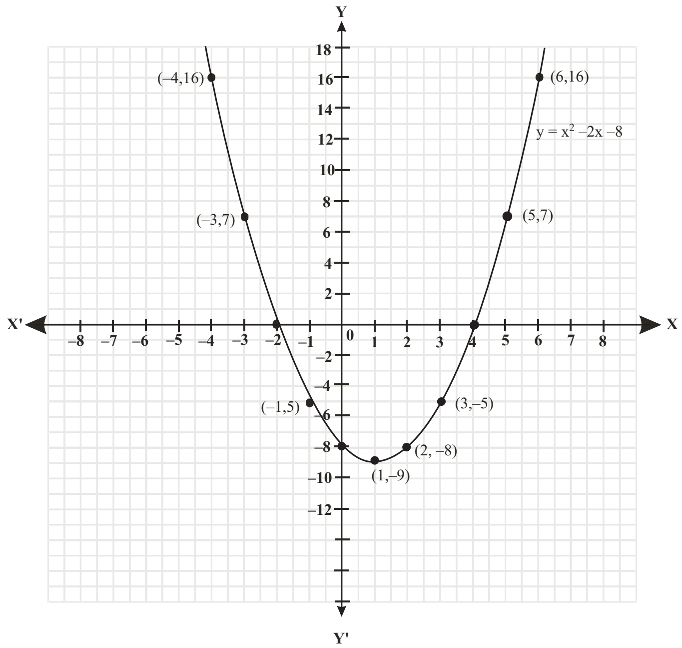
SOLUTION :

Let $y = x^2 - 2x - 8$

The following table gives the values of y or $f(x)$ for various values of x.

x	– 4	– 3	– 2	– 1	0	1	2	3	4	5	6
y	16	7	0	– 5	– 8	– 9	– 8	– 5	0	7	16

Let us now plot the points (– 4, 16), (–3, 7), (–2, 0), (–1, –5), (0, –8), (1, –9), (2, –8), (3, –5), (4, 0), (5, 7) and (6, 16) on a graph paper and draw a smooth free hand curve passing through these points. The curve thus obtained represents the graph of the polynomial $f(x) = x^2 - 2x - 8$ which is known as a parabola.



Note that, on both the axis, scale may be different.

The parabola intersects the x-axis at $x = -2$ and 4. Hence, zeroes or roots of the polynomial are –2 and 4.

ILLUSTRATION : 6

Draw the graph of the quadratic polynomial $f(x) = 3 - 2x - x^2$. Also find its zeroes.

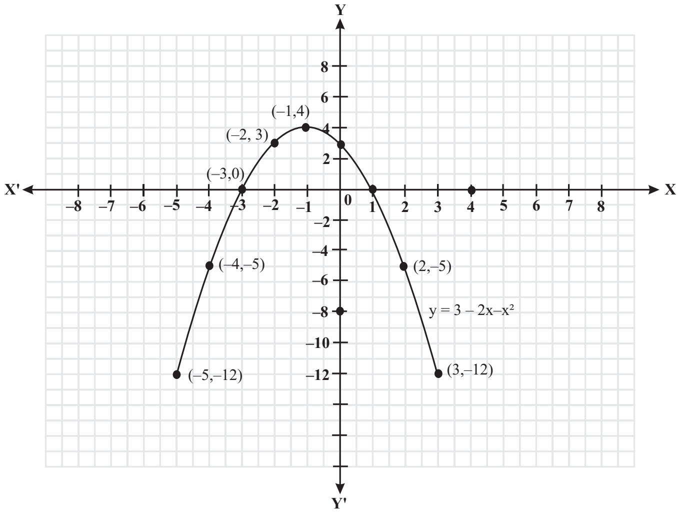
SOLUTION :

Let $y = f(x)$ or $y = 3 - 2x - x^2$

Let us list a few values of y corresponding to a few values of x as follows :

x	– 5	– 4	– 3	– 2	– 1	0	1	2	3
y	–12	– 5	0	3	4	3	0	– 5	– 12

Thus, the following points lie on the graph of polynomial $y = 3 - 2x - x^2$:
(–5, –12), (–4, –5), (–3, 0), (–2, 3), (–1, 4), (0, 3), (1, 0), (2, –5), and (3, – 12)
Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graph of $y = 3 - 2x - x^2$. The curve thus obtained is a parabola.



The parabola intersects x-axis at $x = -3$ and 1. Therefore, zeroes or roots of the polynomial are –3 and 1.

GRAPHS OF A CUBIC POLYNOMIAL

A cubic polynomial is a function of the form $y = ax^3 + bx^2 + cx + d$ where a, b, c & d are real constants but $a \neq 0$.
If $a > 0$, then graph of a cubic function looks similar to one of the graphs in Figures (i), (ii) and (iii).

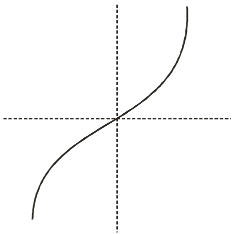


fig. (i)

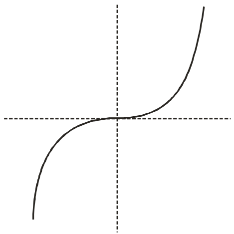


fig. (ii)

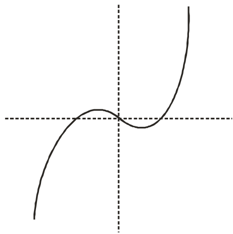


fig. (iii)

If $a < 0$, then graph of the cubic function looks similar to one of the graphs in Figures (iv), (v) and (vi).

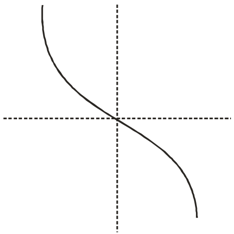


fig. (iv)

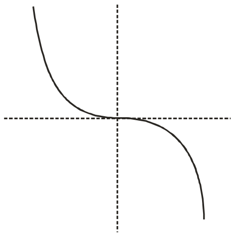


fig. (v)

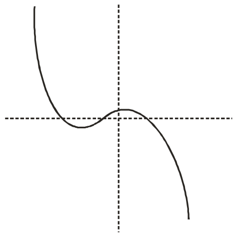


fig. (vi)

ILLUSTRATION : 7

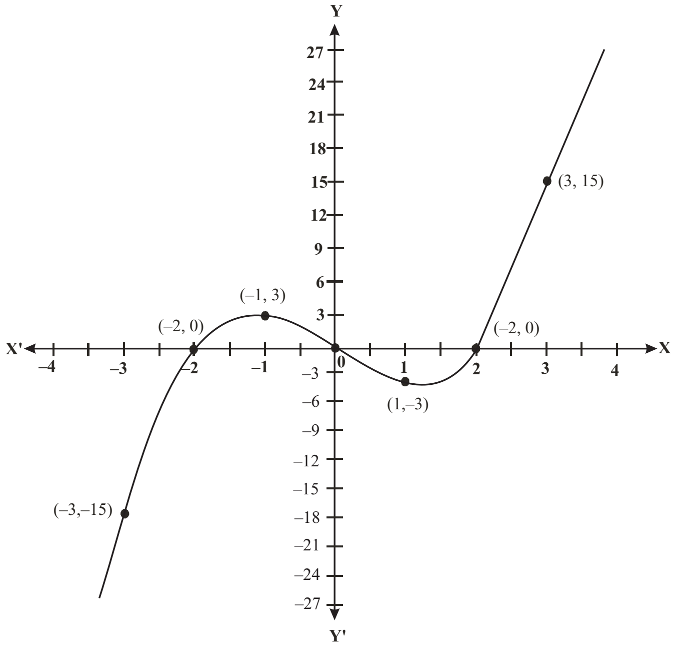
Draw the graph of the polynomial $f(x) = x^3 - 4x$. Also find its zero(es).

SOLUTION :

Let $y = f(x) = x^3 - 4x$.

The values of y for few values of x are listed in the following table :

x	– 3	– 2	– 1	0	1	2	3
$y = x^3 - 4x$	– 15	0	3	0	– 3	0	15



Since the graph of the given polynomial intersects x-axis at –2, 0 and 2. Therefore, zeroes of the given cubic polynomial are –2, 0 and 2.

Learn More

Cubic polynomial has either all roots are rational or one is rational and two are irrational. Irrational roots always occur in conjugate pairs.

RELATIONSHIP BETWEEN ZERO(ES) AND COEFFICIENT OF A POLYNOMIAL

(i) Zero of a linear polynomial $ax + b$, is $x = -\frac{b}{a}$ or $x = -\frac{\text{constant term}}{\text{coefficient of } x}$

(ii) If quadratic polynomial $ax^2 + bx + c = k(x - \alpha)(x - \beta)$, where k is any real constant; then α and β are zeroes of quadratic polynomial $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$.

$$\alpha + \beta = -\frac{b}{a}, \text{ i.e., sum of zeroes} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2} \quad \text{and} \quad \alpha.\beta = -\frac{c}{a} \text{ i.e., product of zeroes} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

We can obtain the quadratic polynomial $p(x)$ as $p(x) = k\,[x^2 - (\text{sum of zeroes})\,x + (\text{product of zeroes})]$

(iii) If cubic polynomial $ax^3 + bx^2 + cx + d = k(x - \alpha)(x - \beta)(x - \gamma)$, where k is any real constant; then α, β and γ are zeroes of cubic polynomial $ax^3 + bx^2 + cx + d$ where $a, b, c, d \in$ R and a $\neq 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \quad \text{and} \quad \alpha\beta\gamma = -\frac{d}{a}$$

ILLUSTRATION : 8

If zeroes of the polynomial $x^2 + 4x + 2a$ are α and $\dfrac{2}{\alpha}$, then find the value of α.

SOLUTION :

Given, α and — are the zeroes of $x^2 + 4x + 2a$.

Now, product of zeroes $= \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

$$\Rightarrow \alpha \times \frac{2}{\alpha} = 2a \Rightarrow 2 = 2\alpha \Rightarrow \alpha = 1$$

Some useful formulas :

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

(ii) $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta) = \alpha+\beta\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(iii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(v) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

DIVISION OF POLYNOMIALS

On dividing a polynomial $p(x)$ by a polynomial $d(x)$, let the quotient be $q(x)$ and the remainder be $r(x)$, then $p(x) = d(x).\ q(x) + r(x)$, where either $r(x) = 0$ or deg. $r(x) <$ deg.$d(x)$

Here, Dividend $= p(x)$, Divisor $= d(x)$, Quotient $= q(x)$ and Remainder $= r(x)$.

Division algorithm of a polynomial by a polynomial :

Step 1 : Arrange the terms of the dividend and the divisor in descending order of their degrees.

Step 2 : Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

Step 3 : Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.

Step 4 : Consider the remainder as new dividend and proceed as before.

Step 5 : Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

Learn More

- **Factor Theorem**
  (i) If $(x - r)$ is a factor of polynomial $f(x)$, then r is a root of polynomial equation $f(x) = 0$.
  (ii) Conversely, if the polynomial equation $f(x) = 0$ has a root r, then $(x - r)$ is factor of polynomial $f(x)$.
- **Remainder Theorem:**
  Let $p(x)$ be any polynomial of degree greater than or equal to one and 'a' be any real number. If $p(x)$ is divided by linear polynomial $(x - a)$, then remainder is $p(a)$.

ILLUSTRATION : 9

Divide $x^3 + x^2 - 2x - 30$ by $x - 3$

SOLUTION :

$$\begin{array}{r|l|l}
x-3 & x^3 + x^2 - 2x - 30 & x^2 + 4x + 10 \\
 & \underset{-\quad +}{x^3 - 3x^2} & \\
\hline
 & 4x^2 - 2x - 30 & \\
 & \underset{-\quad +}{4x^2 - 12x} & \\
\hline
 & 10x - 30 & \\
 & \underset{-\quad +}{10x - 30} & \\
\hline
 & \times & \\
\hline
\end{array}$$

$\therefore (x^3 + x^2 - 2x - 30) \div (x - 3) = x^2 + 4x + 10$

ILLUSTRATION : 10

Find the remainder obtained on dividing $x^3 + 3$ by $x + 1$.

SOLUTION :

$$
\begin{array}{r|l}
x+1 & x^3 + 3 \;(\; x^2 - x + 1 \\
 & x^3 + x^2 \\
 & - \quad - \\
\hline
 & -x^2 \quad + 3 \\
 & -x^2 - x \\
 & + \quad + \\
\hline
 & x + 3 \\
 & x + 1 \\
 & - \quad - \\
\hline
 & 2
\end{array}
$$

Hence, remainder = 2

Alternate Method : To find remainder we can use remainder theorem.

Let $f(x) = x^3 + 3$

Put Divisor = 0

$\Rightarrow x + 1 = 0 \quad \Rightarrow x = -1$

Then remainder = $f(-1) = (-1)^3 + 3 = -1 + 3 = 2$

So, 2 is the remainder.

ILLUSTRATION : 11

Find all the zero(es) of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zero(es) are $\sqrt{2}$ and $-\sqrt{2}$.

SOLUTION :

Two zeroes are $\sqrt{2}$ and $-\sqrt{2}$. Hence, $(x-\sqrt{2})$ and $(x+\sqrt{2})$ are the factors of given polynomial.

$\Rightarrow (x-\sqrt{2})(x+\sqrt{2}) = x^2 - 2$, is a factor of the given polynomial.

Now, we divide the given polynomial by $x^2 - 2$

$$
\begin{array}{r|l}
 & 2x^2 - 3x + 1 \\
\hline
x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \\
 & 2x^4 \qquad\quad - 4x^2 \\
 & - \qquad\qquad + \\
\hline
 & -3x^3 + x^2 + 6x - 2 \\
 & -3x^3 \qquad + 6x \\
 & + \qquad\quad - \\
\hline
 & x^2 \qquad - 2 \\
 & x^2 \qquad - 2 \\
 & - \qquad + \\
\hline
 & 0
\end{array}
$$

(First term of quotient is $\frac{2x^4}{x^2} = 2x^2$)

(Second term of quotient is $\frac{-3x^3}{x^2} = -3x$)

(Third term of quotient is $\frac{x^2}{x^2} = 1$)

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$.
Now, by splitting $-3x$, we factorise $2x^2 - 3x + 1$.

$$
\begin{aligned}
2x^2 - 3x + 1 &= 2x^2 - 2x - x + 1 \\
&= 2x(x-1) - 1(x-1) \\
&= (x-1)(2x-1)
\end{aligned}
$$

So, the remaining two zero(es) are given by $x = \frac{1}{2}$ and $x = 1$. Therefore, the zero(es) of the given polynomial are $\sqrt{2}, -\sqrt{2}, \frac{1}{2}$ and 1.

CASE STUDY : Division Algorithm, Factor Theorem, Remainder Theorem and zero(es) of the Polynomial

- **Division Algorithm :**
  (Dividend) = (Quotient) × (Divisor) + (Remainder)
- **Factor Theorem :**
  $(x - a)$ is a factor of polynomial $f(x)$, if $f(a) = 0$
  $(x + a)$ is a factor of polynomial $f(x)$, if $f(-a) = 0$
  $(ax - b)$ is a factor of polynomial $f(x)$, if $f\left(\frac{b}{a}\right) = 0$
  $(ax + b)$ is a factor of polynomial $f(x)$, if $f\left(\frac{-b}{a}\right) = 0$
- **Remainder Theorem :**
  If polynomial $f(x)$ is divided by $(x - a)$, then remainder = $f(a)$
  If polynomial $f(x)$ is divided by $(x + a)$, then remainder = $f(-a)$
  If polynomial $f(x)$ is divided by $(ax - b)$, then remainder = $f\left(\frac{b}{a}\right)$
  If polynomial $f(x)$ is divided by $(ax + b)$, then remainder = $f\left(-\frac{b}{a}\right)$
- **Zeroes of a Polynomial**
  A number α is a zero of polynomial $f(x)$, if $f(\alpha) = 0$.
  Number of zero(s) of a polynomial = Degree of the polynomial.
  Hence number zero(s) of a linear polynomial, quadratic polynomial, cubic polynomial and biquadratic polynomial are 1, 2, 3 and 4 respectively.

CASE-II : To find the quotient and remainder using division algorithm

Illustration: *Apply the division algorithm to find the quotient and remainder on dividing $f(x) = x^3 - 6x^2 + 11x - 6$, by $g(x) = x + 2$.*

Sol. We have,

$f(x) = x^3 - 6x^2 + 11x - 6$ and $g(x) = x + 2$

We find that degree f(x) = 3 and degree g(x) = 1. Therefore, quotient q(x) is of degree 3 – 1 = 2 and the remainder r(x) is of degree zero.

Let $q(x) = ax^2 + bx + c$ and $r(x) = k$.

By using division algorithm, we obtain

$f(x) = q(x) \times g(x) + r(x)$

$\Rightarrow\ x^3 - 6x^2 + 11x - 6 = (ax^2 + bx + c)(x + 2) + k$

$\Rightarrow\ x^3 - 6x^2 + 11x - 6 = ax^3 + (2a + b)x^2 + (2b + c)x + (2c + k)$

Equating the coefficients of like powers of x and can start term on both sides, we get

$1 = a$ [On equating the coefficients of x^3]

$-6 = 2a + b$ [On equating the coefficients of x^2]

$11 = 2b + c$ [On equating the coefficients of x]

and, $-6 = 2c + k$ [On equating the coefficients terms]

Solving these equations, we get $a = 1$, $b = -8$, $c = 27$ and $k = -60$

$\therefore$ Quotient $q(x) = ax^2 + bx + c = x^2 - 8x + 27$ and, Remainder $r(x) = k = -60$.

CASE-I : To find the remaining zeroes of a polynomial when some of its zeroes are given

Illustration: *Obtain all the zeroes of the polynomial $f(x) = 3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.*

Sol. It is given that $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$ are two zeroes of $f(x)$. Therefore, $\left(x-\sqrt{\frac{5}{3}}\right)\left(x+\sqrt{\frac{5}{3}}\right)$ is a factor of $f(x)$. But, $\left(x-\sqrt{\frac{5}{3}}\right)\left(x+\sqrt{\frac{5}{3}}\right)$

$= \left(x^2 - \frac{5}{3}\right) = \frac{1}{3}(3x^2 - 5)$. Therefore, $(3x^2 - 5)$ is a factor of f(x).

Let us now divide $f(x)$ by $3x^2 - 5$. Using long division method, we obtain

$$
\begin{array}{r|l}
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \ \ (x^2 + 2x + 1 \\
 & \underline{\overset{-}{3x^4} \qquad \overset{+}{-5x^2}} \\
 & 6x^3 + 3x^2 - 10x - 5 \\
 & \underline{\overset{-}{6x^3} \qquad \overset{+}{-10x}} \\
 & 3x^2 - 5 \\
 & \underline{\overset{-}{3x^2} \overset{+}{-5}} \\
 & 0
\end{array}
$$

Clearly, Quotient = $x^2 + 2x + 1$ and Remainder = 0.

By division algorithm, we obtain

$f(x) = (3x^2 - 5)(x^2 + 2x + 1) + 0$

$\Rightarrow f(x) = (\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5})(x + 1)^2$

Thus, the factors of $f(x)$ are $\sqrt{3}x + \sqrt{5}$, $\sqrt{3}x - \sqrt{5}$, $x + 1$ and $x + 1$.

Equating each factor to zero, we obtain $x = -\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}, -1, -1$. Hence, the zeroes of f(x) are $-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}, -1$ and -1.

CASE-III : To check whether a given polynomial in a factor of the other polynomial by applying the Division Algorithm

Illustration: *By applying division algorithm prove that the polynomial* $g(x) = x^2 + 3x + 1$ *is a factor of the polynomial* $f(x) = 3x^4 + 5x^3 - 7x^2 + 2x + 2$

Sol. We find that degree of $f(x) = 4$ and degree $g(x) = 2$. Therefore, quotient $q(x)$ is of degree $2(= 4 - 2)$ and the remainder $r(x)$ is of degree 1 or less. Let $q(x) = ax^2 + bx + c$ and $r(x) = px + q$.

Using division algorithm, we have

$$f(x) = g(x) \times q(x) + r(x)$$

$\Rightarrow \quad 3x^4 + 5x^3 - 7x^2 + 2x + 2 = (ax^2 + bx + c)(x^2 + 3x + 1) + (px + q)$

$\Rightarrow \quad 3x^4 + 5x^3 - 7x^2 + 2x + 2 = ax^4 + (3a + b)x^3 + (a + 3b + c)x^2 + (b + 3c + p)x + (c + q)$

Equating coefficients of various powers of x and constant term from both sides, we get

$a = 3$ [On equating the coefficients of x^4]

$3a + b = 5$ [On equating the coefficients of x^3]

$a + c + 3b = -7$ [On equating the coefficients of x^2]

$b + 3c + p = 2$ [On equating the coefficients of x]

and, $c + q = 2$ [On equating the constant terms]

Solving these equations, we get

$a = 3, b = -4, c = 2, p = 0$ and $q = 0$

$\therefore$ Quotient $q(x) = 3x^2 - 4x + 2$ and, Remainder $r(x) = 0x + 0 = 0$

Since, remainder $r(x) = 0$. Hence, $g(x)$ is a factor of $f(x)$.

Think Out of the Box

Q. 1. If $x = \dfrac{1}{2\sqrt{3}}$ is a root of the polynomial $f(x) = 2kx^2 - 7x + k$, then find other root of the polynomial $f(x)$.

[**Hint:** $f\left(\dfrac{1}{2\sqrt{3}}\right) = 0 \Rightarrow k = \sqrt{3}$. **Ans.** $x = \sqrt{3}$]

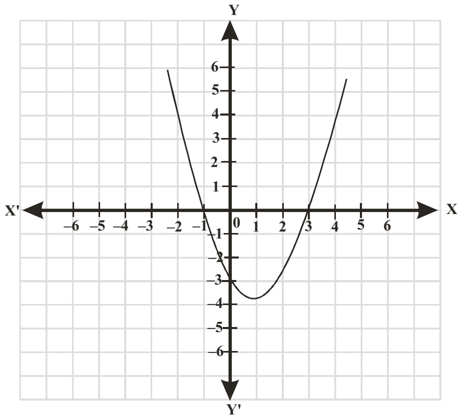
Q. 2. If the quotient obtained on dividing the polynomial $f(x) = 3x^3 + 11x^2 + 34x + 106$ by $g(x) = x - 3$ using division algorithm is $3x^2 + px + q$, then find p, q and also the reminder.
[**Hint:** $f(x) = g(x)(3x^2 + 20x + 94) + 388$. **Ans.** $p = 20$, $q = 94$, remainder $= 388$]

MISCELLANEOUS SOLVED EXAMPLES

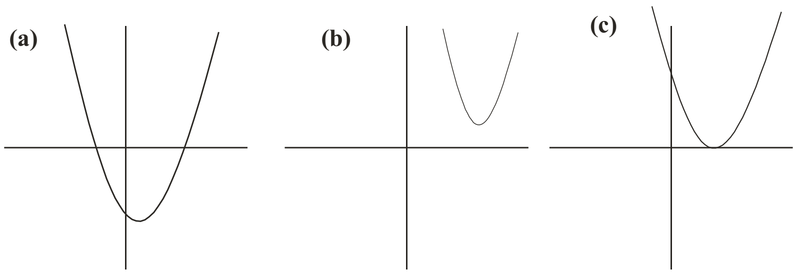
1. $f(x) = x^2 - 2x - 3$. Find the roots of $f(x)$, and sketch the graph of $y = f(x)$.

Sol. $f(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$. Therefore, the roots are -1 and 3.

The graph intersects the X-axis at $x = -1$ and 3.



2. How many real roots, i.e., roots that are real numbers, has the parabola of each graph ?



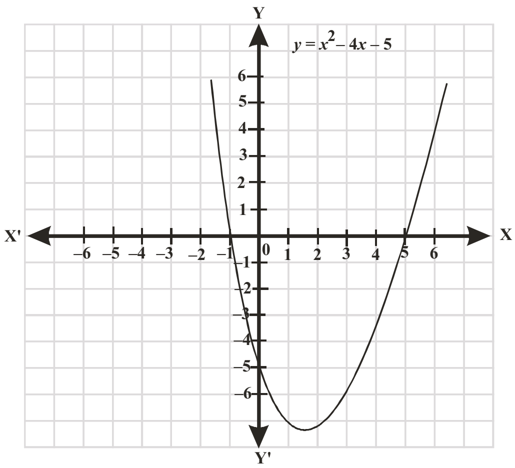
Sol. Graph (a) has two real roots. It has two x-intercepts (i.e., the graph intersects the X-axis at two different points)

Graph (b) has no real roots. It has no x-intercepts. Both roots are complex.

Graph (c) has two real roots which are equal.

3. Inspect the following graph to solve this equation : $x^2 - 4x - 5 = 0$

Sol. The quadratic polynomial will have the value 0 at $x = -1$ and 5.



4. Find the zero(es) of the polynomial $x^2 - 3$ and verify the relationship between the zero(es) and the coefficients.

Sol. Recall the identity $a^2 - b^2 = (a - b)(a + b)$. Using it, we can write:

$x^2 - 3 = \left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$

Now, Sum of zeroes $= \sqrt{3}-\sqrt{3} = 0 = \dfrac{-\text{(Coefficient of } x)}{\text{Coefficient of } x^2}$

Product of zeroes $= \left(\sqrt{3}\right)\left(-\sqrt{3}\right) = -3 = \dfrac{-3}{1} = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

5. Show that $x = 2$ is a root of $2x^3 + x^2 - 7x - 6$

Sol. Let $p(x) = 2x^3 + x^2 - 7x - 6$

$p(2) = 2(2)^3 + (2)^2 - 7(2) - 6 = 16 + 4 - 14 - 6 = 0$

Hence $x = 2$ is a root of $p(x)$.

6. If $x = \dfrac{4}{3}$ is a root of the polynomial $f(x) = 6x^3 - 11x^2 + kx - 20$, then find the value of k.

Sol. Let $f(x) = 6x^3 - 11x^2 + kx - 20$ and if $x = \dfrac{4}{3}$ is root of polynomial, then $f\left(\dfrac{4}{3}\right) = 0$

$\therefore \quad f\left(\dfrac{4}{3}\right) = 6\left(\dfrac{4}{3}\right)^3 - 11\left(\dfrac{4}{3}\right)^2 + k\left(\dfrac{4}{3}\right) - 20 = 0$

$\Rightarrow \quad 6\left(\dfrac{64}{27}\right) - 11\left(\dfrac{16}{9}\right) + \dfrac{4k}{3} - 20 = 0 \Rightarrow 128 - 176 + 12k - 180 = 0 \quad \Rightarrow \quad 12k + 128 - 356 = 0 \Rightarrow 12k = 228 \Rightarrow k = 19$

7. Find the remainder when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $g(x) = 1 - 2x$.

Sol. Put $1 - 2x = 0 \Rightarrow 2x = 1 \Rightarrow x = \dfrac{1}{2}$

Remainder $= f\left(\dfrac{1}{2}\right) = \left(\dfrac{1}{2}\right)^3 - 6\left(\dfrac{1}{2}\right)^2 + 2\left(\dfrac{1}{2}\right) - 4 = \dfrac{1}{8} - \dfrac{3}{2} + 1 - 4 = \dfrac{1-12+8-32}{8} = -\dfrac{35}{8}$

8. The polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by $x + 2$ if the remainder in each case is the same, find the value of a.

Sol. Let $p(x) = ax^3 + 3x^2 - 13$ and $q(x) = 2x^3 - 5x + a$

When $p(x)$ and $q(x)$ are divided by $x + 2 = 0 \Rightarrow x = -2$

$\therefore \; p(-2) = q(-2) \Rightarrow a(-2)^3 + 3(-2)^2 - 13 = 2(-2)^3 - 5(-2) + a \Rightarrow -8a + 12 - 13 = -16 + 10 + a$

$\Rightarrow -9a = -5 \Rightarrow a = \dfrac{5}{9}$

9. Find α and β if $x + 1$ and $x + 2$ are factors of $p(x) = x^3 + 3x^2 - 2\alpha x + \beta$

Sol. Put $x + 1 = 0$ or $x = -1$ and $x + 2 = 0$ or $x = -2$ in $p(x)$

Then, $p(-1) = 0$ and $p(-2) = 0$

$\Rightarrow \; p(-1) = (-1)^3 + 3(-1)^2 - 2\alpha(-1) + \beta = 0 \quad \Rightarrow -1 + 3 + 2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha - 2$ ... (i)

$p(-2) = (-2)^3 + 3(-2)^2 - 2\alpha(-2) + \beta = 0 \quad \Rightarrow -8 + 12 + 4\alpha + \beta = 0 \Rightarrow \beta = -4\alpha - 4$ ... (ii)

By equalising equations (i) and (ii), we get

$-2\alpha - 2 = -4\alpha - 4 \Rightarrow 2\alpha = -2 \Rightarrow \alpha = -1$

put $\alpha = -1$ in eq. (i) $\Rightarrow \beta = -2(-1) - 2 = 2 - 2 = 0$ Hence, $\alpha = -1, \; \beta = 0$

10. What must be added to $3x^3 + x^2 - 22x + 9$, so that the result is exactly divisible by $3x^2 + 7x - 6$.

Sol. Let $p(x) = 3x^3 + x^2 - 22x + 9$ and $r(x) = 3x^2 + 7x - 6$

We know that if $p(x)$ is not exactly divisible by $q(x)$, then the remainder be $r(x)$ and degree of $r(x)$ is less than $q(x)$ or divisor.