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Chapter

0

Basic Mathematics for Physics

INTRODUCTION

If you want to excel in science field, it is very important to have good mathematical aptitude. This chapter is presenting Basic Mathematics that is very useful for developing problem solving strategies whenever required. It includes application in physics section that will develop your scientific temper. Go through it twice or thrice before start learning physics. Take your teachers help wherever you feel any doubt or require more illustrations. "Mathematics is the language of physics".

TRIGONOMETRIC IDENTITIES

1. $\sin(-\theta) = -\sin\theta$
2. $\cos(-\theta) = \cos\theta$
3. $\tan(-\theta) = -\tan\theta$
4. $(\sin\theta) / (\cos\theta) = \tan\theta$
5. $\sin 2\theta = 2\sin\theta\cos\theta$
6. $\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta$
7. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
8. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$; If $\alpha = 90^\circ$, $\sin(90^\circ \pm \beta) = \cos\beta$; If $\alpha = \beta$, $\sin 2\beta = 2\sin\beta\cos\beta$
9. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$;

   If $\alpha = 90^\circ$, $\cos(90^\circ \pm \beta) = \mp \sin$; If $\alpha = \beta$, $\cos 2\beta = \cos^2\beta - \sin^2\beta = 1 - 2\sin^2\beta$.
10. $\sin(90^\circ - \theta) = \cos\theta$; $\cos(90^\circ - \theta) = \sin\theta$; $\tan(90^\circ - \theta) = \cot\theta$
11. $\sin(90^\circ + \theta) = \cos\theta$; $\cos(90^\circ + \theta) = -\sin\theta$; $\tan(90^\circ + \theta) = -\cot\theta$
12. $\sin(180^\circ - \theta) = \sin\theta$; $\cos(180^\circ - \theta) = -\cos\theta$; $\tan(180^\circ - \theta) = -\tan\theta$
13. $\sin(180^\circ + \theta) = -\sin\theta$; $\cos(180^\circ + \theta) = -\cos\theta$; $\tan(180^\circ + \theta) = \tan\theta$
14. $\sin(270^\circ - \theta) = -\cos\theta$; $\cos(270^\circ - \theta) = -\sin\theta$; $\tan(270^\circ - \theta) = \cot\theta$
15. $\sin(270^\circ + \theta) = -\cos\theta$; $\cos(270^\circ + \theta) = \sin\theta$; $\tan(270^\circ + \theta) = -\cot\theta$
16. $\sin(360^\circ - \theta) = -\sin\theta$; $\cos(360^\circ - \theta) = \cos\theta$; $\tan(360^\circ - \theta) = -\tan\theta$
17. $\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta - \tan^2\theta = 1$; $\csc^2\theta - \cot^2\theta = 1$

18. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

19. $\sin A + \sin B = 2\sin\dfrac{(A+B)}{2}\cos\dfrac{(A-B)}{2}$

20. $\cos A - \cos B = 2\cos\dfrac{(A+B)}{2}\cos\dfrac{(A-B)}{2}$

21. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

22. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \pm \tan A \tan B}$

23. $\sin A - \sin B = 2\cos\dfrac{(A\quad B)}{}\cos\dfrac{(A-B)}{2}$

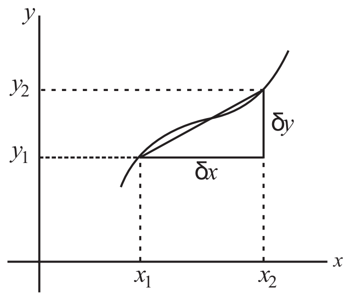
24. $\cos A - \cos B = -2\sin\dfrac{(A+B)}{2}\sin\dfrac{(A-B)}{2}$

25. $1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$

DIFFERENTIAL CALCULUS

The derivative y with respect to x is defined as the limit of the slopes of chords drawn between two points on the y versus x curve as Δx approaches zero. Mathematically, we write this defintion as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{y(x+\Delta x) - y(x)}{\Delta x} \qquad ...(1)$$



where Δy and Δx are defined as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (See figure)

A useful expression to remember when $y(x) = ax^n$, where a is a constant and n is any positive or negative number (integer or fraction), is

$$\frac{dy}{dx} = nax^{n-1} \qquad ...(2)$$

If $y(x)$ is a polynomial or algebraic function of x, we apply above equation to each term in the polynomial and take $\dfrac{da}{dx} = 0$. It is important to note that $\dfrac{dy}{dx}$ does not mean dy divided by dx, but is simply a notation of the limiting process of the derivative. In Examples, we evaluate the derivatives of several well-behaved functions.

ILLUSTRATION : 1

Solve $y(x) = 8x^5 + 4x^3 + 2x + 7$

SOLUTION:

Applying equation 2 to each term independently, and remembring that $\dfrac{d}{dx}$ (constant) = 0, we have

$$\frac{dy}{dx} = 8(5)x^4 + 4(3)x^2 + 2(1)x^0 + 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 40x^4 + 12x^2 + 2$$

Special Properties of the Derivative

(i) **Derivative of the product of two functions:** If a function y is given by the product of two functions, say $g(x)$ and $h(x)$, then the derivative of y is defined as

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x)h(x)] = g\frac{dh}{dx} + h\frac{dg}{dx} \qquad ...(3)$$

(ii) **Derivative of the sum of two functions:** If a function y is equal to the sum of two functions, then the derivative of the sum is equal to the sum of the derivatives.

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x) + h(x)] = \frac{dg}{dx} + \frac{dh}{dx} \qquad ...(4)$$

(iii) **Chain rule of differential calculus:** If $y = f(x)$ and x is a function of some other variable z, then $\frac{dy}{dx}$ can be written as the product of two derivatives.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad ...(5)$$

(iv) **The second derivative:** The second derivative of y with respect to x is defined as the derivative of the function $\frac{dy}{dx}$ (or the derivative of the derivative). It is usually written

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad ...(6)$$

Derivatives for Several Functions

(i)	$\frac{d}{dx}(a) = 0$ where 'a' is a constant	(vi)	$\frac{d}{dx}(\tan ax) = a\sec^2 ax$
(ii)	$\frac{d}{dx}(ax^n) = nax^{n-1}$	(vii)	$\frac{d}{dx}(\cot ax) = -a\ \text{cosec}^2 ax$
(iii)	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	(viii)	$\frac{d}{dx}(\sec x) = \tan x \sec x$
(iv)	$\frac{d}{dx}(\sin ax) = a\cos ax$	(ix)	$\frac{d}{dx}(\text{cosec}\, x) = -\cot x\, \text{cosec}\, x$
(v)	$\frac{d}{dx}(\cos ax) = -a\sin ax$	(x)	$\frac{d}{dx}(\ln ax) = \frac{1}{x}$

Maxima and Minima

To find the maxima and minima for a function or extreme values of the function $f(x)$ we proceed as follows.

Step-I : Find $f'(x) = \frac{d}{dx}[f(x)]$

Step-II : Put $f'(x) = 0$ and solve the obtained equations for values of x or simply saying, find the zero of the function $f'(x)$. Let $x = a_1, a_2, ...\ a_n$ be the zeros of the function $f'(x)$ also called the stationary values of $f'(x)$.

Step-III : Find $f''(x) = \frac{d^2}{dx^2}[f(x)] = \frac{d}{dx}[f'(x)]$

Step-IV : Find the values of $f''(x)$ at $x = a_1, a_2, ...a_n$.

Step-V : If $f''(x) > 0$ then we get MINIMA
If $f''(x) < 0$ then we get MAXIMA

Step-VI : If $f''(x) = 0$ then we get the point of INFLEXION i.e. a point which is neither MINIMA nor MAXIMA.

INTEGRAL CALCULUS

Integration is the inverse of differentiation.

Some indefinite integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1}$ (provided $n \neq -1$)

(ii) $\int \frac{dx}{x} = \int x^{-1} dx = \ln x$

(iii) $\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$

(iv) $\int \frac{dx}{(a+bx)^2} = \frac{1}{b(a+bx)}$

(v) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

(vi) $\int e^{ax} dx = \frac{1}{a}e^{ax}$

(vii) $\int \sin ax dx = -\frac{1}{a}\cos ax$

(viii) $\int \cos ax dx = \frac{1}{a}\sin ax$

ILLUSRATION : 2

Integrate the following functions with respect to *x*

(a) $\int (5x^2 + 3x - 2)dx$ **(b)** $\int \left(4\sin x - \frac{2}{x}\right)dx$ **(c)** $\int \frac{dx}{4x+5}$ **(d)** $\int (6x+2)^3 dx$

SOLUTION :

(a) $\int (5x^2 + 3x - 2)dx = 5\int x^2 dx + 3\int x\, dx - 2\int dx = \frac{5x^2}{3} + \frac{3x^2}{2} - 2x + c$

(b) $\int \left(4\sin x - \frac{2}{x}\right)dx = 4\int \sin x\, dx - 2\int \frac{dx}{x} = -4\cos x - 2\ln x + c$

(c) $\int \frac{dx}{4x+5} = \frac{1}{4}\int \frac{dX}{X}$, where $X = 4x + 5$

$= \frac{1}{4}\ln X + c_1 = \frac{1}{4}\ln(4x+5) + c_2$

(d) $\int (6x+2)^3 dx = \frac{1}{6}\int X^3 dX$, where $X = 6x + 2$

$= \frac{1}{6}\left(\frac{X^4}{4}\right) + c_1 = \frac{(6x+2)^4}{24} + c_2$

LOGARITHMS

(i) $e \approx 2.7183$

(ii) If $e^x = y$, then $x = \log_e y = \ln y$

(iii) If $10^x = y$, then $x = \log_{10} y$

(iv) $\log_{10} y = 0.4343 \log_e y = 0.4343 \ln y$

(v) $\log(ab) = \log(a) + \log(b)$

(vi) $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

(vii) $\log a^n = n\log(a)$

VECTORS

Physical quantities which need both magnitude and direction to express them and obey the triangle or parallelogram law of vector addition are called vectors.

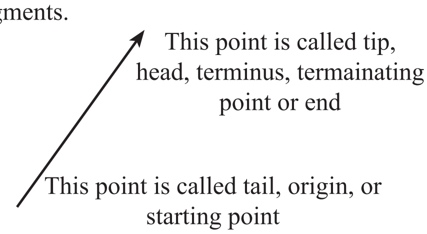
Examples: Displacement, velocity, acceleration, momentum, force etc.

Representation of Vector

Vectors are represented either graphically or by symbols.

Graphical representation of vectors

Vectors are represented by directed line segments.



The length of the line segment represents the magnitude of the vector and its arrow-head depicts its direction. The arrow head can be put on the extreme end of the line or at any other point of the line.

* represents a vector whose direction is perpendicular to the plane of the paper, coming out of the paper i.e. towards you. Similarly, ⊗ represents a vector which is perpendicular to the plane of the paper but going inside it.*

Symbolic Representation of Vectors

1. Vectors are represented in either of the following two ways.
Either as $\vec{r}$, $\overrightarrow{PQ}$, etc.i.e., arrow on the head.
or as **r, PQ**, etc. i.e., bold letters.

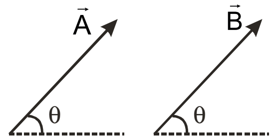
2. Magnitude (also called modulus or mod values) of vectors are represented by normal alphabets.
e.g. magnitude of $\vec{r}$ or $\overrightarrow{PQ}$ = r or $|\vec{r}|$, PQ or $|\overrightarrow{PQ}|$

3. Unit vectors (detail study in types of vector) (i.e. whose magnitude is one unit) are represented as $\hat{r}, \widehat{PQ}$ etc. (read as r-cap, PQ-cap, r-hat, PQ-hat or as r-caret, PQ-caret etc.)

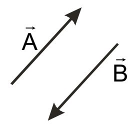
Types of Vector

1. Equal vectors : Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal when they have equal magnitudes and same direction.

2. Parallel vectors : Two vectors $\vec{A}$ and $\vec{B}$ are said to be parallel when
(i) both have same direction.
(ii) one vector is scalar (positive non-zero) multiple of another vector.
Use : If a vector is shifted parallel to itself there will be no change in it.



3. Anti-parallel vectors : Two vectors $\vec{A}$ and $\vec{B}$ are said to be anti-parallel when
(i) both have opposite direction.
(ii) one vector is scalar (non-zero negative) multiple of another vector.
Use : In subtraction of vector.



4. Collinear vectors : When the vectors under consideration can share the same support or have a common support then the considered vectors are collinear vectors.

5. Zero or null vector $(\vec{0})$: A vector having zero magnitude and arbitrary direction (not known to us) is a zero vector.
Properties of zero vector :
(i) The sum of a finite vector $\vec{A}$ and the zero vector is equal to the finite vector.
i.e., $\vec{A} + \vec{0} = \vec{A}$
(ii) The multiplication of a zero vector by a finite number n is equal to the zero vector.
i.e., $n\,\vec{0} = \vec{0}$
(iii) The multiplication of a finite vector $\vec{A}$ by zero is equal to zero vector.
i.e., $0\,\vec{A} = \vec{0}$
Examples : The position vector of the origin in a system of coordinates is a zero vector. If an object is stationary, then its displacement in a finite time is a zero vector.

6. Unit vector : A vector divided by its magnitude (= 1)is a unit vector. Unit vector for $\vec{A}$ is $\hat{A}$ (read as A cap or A hat)
Since, $\hat{A} = \dfrac{\vec{A}}{A} \Rightarrow \vec{A} = A\hat{A}$ thus we can say that unit vector gives us the direction.
Orthogonal unit vectors : $\hat{i}, \hat{j}$ and $\hat{k}$ are called orthogonal unit vectors. These vectors must form a right handed triad (It is a coordinate system such that when we curl the fingers of right hand from x to y then we must get the direction of z along thumb). Thus $\hat{i} = \dfrac{\vec{x}}{x},\ \hat{j} = \dfrac{\vec{y}}{y},\ \hat{k} = \dfrac{\vec{z}}{z}$ $\therefore$ $\vec{x} = x\hat{i},\ \vec{y} = y\hat{j}$ and, $\vec{z} = z\hat{k}$
Use : To convert any magnitude into vector form multiply it by unit vector in that direction.

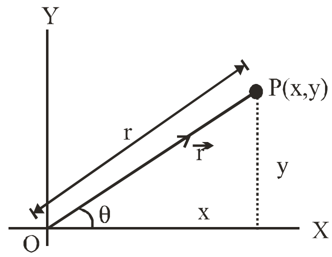
7. Coplanar vector : Three or more vectors are called coplanar vector if they lie in the same plane. Two (free) vectors are always coplanar.

8. **Free and localised vectors :** A vector whose value depends only on its length and direction and is independent of its position in space, is called a free vector. Such a vector may be shifted, at pleasure, parallel to itself. However, if a vector is associated (bound) with a given line or point, it is said to be localised in that line or at that point.

9. **Coterminous vectors :** Vectors having the same terminating point are called coterminous vectors.

10. **Area vector :** The area-vector of a plane figure is a vector whose magnitude is equal to the area of the plane figure and whose direction is perpendicular to the plane area, as given by R.H. Screw rule. (We will learn rule in vector product)

11. **Position vector $(\vec{r})$:** It specifies the position of a point with reference to the given set of coordinate axes. If we join the origin O with the given point P, then $\overrightarrow{OP}$ is called position vector of P.



It is generally represented by $\vec{r} \cdot \vec{r}$ means the magnitude as well as θ i.e., the direction of the point P.

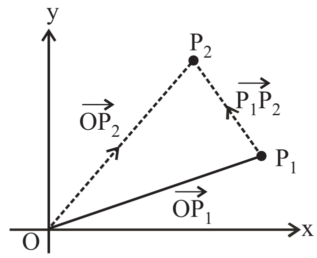
$\therefore \quad r = |\vec{r}| = \sqrt{x^2 + y^2} \quad$ and $\quad \tan\theta = \dfrac{y}{x}$

12. **Displacement vector $(\vec{s})$:** If a particle moves from position P_1 to position P_2, we call $\overrightarrow{P_1P_2}$ as displacement vector. Suppose the particle was originally at the origin O. It first moves to P_1, thereafter to P_2, then $\overrightarrow{OP_2}$ is called final displacement vector.

Symbolically, we state it as $\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$ or $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

or, Displacement vector = final position of vector – initial position of vector.

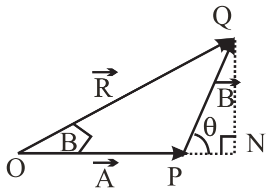
or, Displacement = final position – initial position



LAWS OF VECTOR ADDITION

Triangle Law of Vector Addition

It states that if two vectors acting on a particle at the same time are represented in magnitude and direction by the two sides of a triangle taken in one order, their resultant vector is represented in magnitude and direction by the third side of the triangle taken in opposite order.

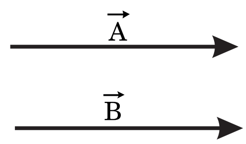


Magnitude of $\vec{R}$ is given by $\quad R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

Direction of $\vec{R}$: Let the resultant $\vec{R}$ make an angle β with the direction of $\vec{A}$, then from right angled triangle QNO,

$$\tan\beta = \frac{QN}{ON} = \frac{QN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta}$$



(i) $|\vec{R}|$ is maximum, if $\cos\theta = 1$ i.e., $\theta = 0°$ (parallel vectors)

$R_{max} = \sqrt{A^2 + B^2 + 2AB} = A + B$

(ii) $|\vec{R}|$ is minimum, if $\cos\theta = 1$ i.e., $\theta = 180°$ (opposite vectors)

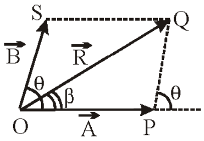
$R_{min} = \sqrt{A^2 + B^2 - 2AB} = A - B$

(iii) If the vectors A and B are orthogonal, i.e. $\theta = 90°$, then, $R = \sqrt{A^2 + B^2}$

Parallelogram Law of Vector Addition

It states, that if two vectors are represented in magnitude and direction by the two adjacent sides of a parallelogram then their resultant is represented in magnitude and direction by the diagonal of the parallelogram.

Let the two vectors $\vec{A}$ and $\vec{B}$, inclined at angle θ are represented by sides $\overrightarrow{OP}$ and $\overrightarrow{OS}$ of parallelogram OPQS, then resultant vector $\vec{R}$ is represents by diagonal $\overrightarrow{OQ}$ of the parallelogram.



$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$; and $\tan\beta = \dfrac{B\sin\theta}{A + B\cos\theta}$

If $\theta < 90^o$ i.e., acute angle then, $\vec{R} = \vec{A} + \vec{B}$ and $\vec{R}$ is called major or main diagonal of parallelogram.

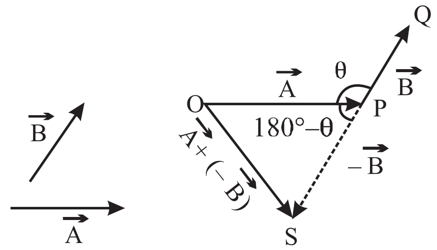
If $\theta < 90^o$ i.e., obtuse angle then, $\vec{R} = \vec{A} + \vec{B}$ and $\vec{R}$ is called minor diagonal of the parallelogram.

SUBTRACTION OF VECTOR

Subtraction of a vector $\vec{B}$ from a vector $\vec{B}$ is defined as the addition of vector $-\vec{B}$ (negative of vector $\vec{B}$) to vector $\vec{A}$.

Thus $\vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$

If θ is the angle between $\vec{A}$ an $\vec{B}$, then angle between $\vec{A}$ and $-\vec{B}$ is $(180^o - \theta)$.



(a) (b)

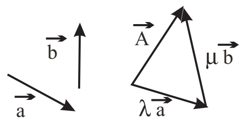
(a) Addition of a vector to its own negative vector or null vector $\vec{A} + (-\vec{A}) = \vec{O}$ i.e., a vector with zero magnitude and an arbitrary direction.
(b) Subtraction is not commutative, i.e., $\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$
(c) Change in a vector physical quantity means subtraction of initial vector from the final vector.

COMPONENTS OF A VECTOR AND RESOLUTION OF VECTORS

Components of a Vector

If $\vec{a}$ and $\vec{b}$ be any two non-zero vectors in a plane with different directions and be another vector in the same plane then, $\vec{A}$ can be expressed as a sum of two vectors-one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number.

$\vec{A} = \lambda\vec{a} + \mu\vec{b}$ (where λ and μ are real numbers)



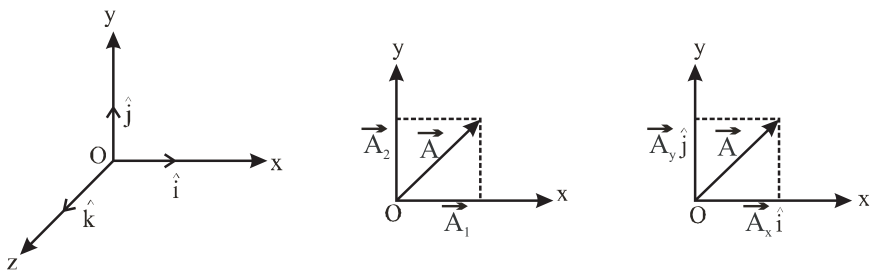
We say that $\vec{A}$ has been resolved into two component vectors $\lambda\vec{a}$ and $\mu\vec{b}$ along $\vec{a}$ and $\vec{b}$ respectively. Hence one can resolve a given vector into two component vectors along a set of two vectors-all the three lie in the same plane.

Resolution of Vectors

The process of splitting a single vector into two or more vectors, which together produce the same effect as is produced by single vector alone, is called resolution of vectors.

The vectors into which the given single vector is splitted are called components of a vector. (Resolution of a vector is just opposite to composition of vectors).

Consider a vector $\vec{A}$ that lies in xy plane, $\vec{A} = \vec{A}_1 + \vec{A}_2$ as shown.

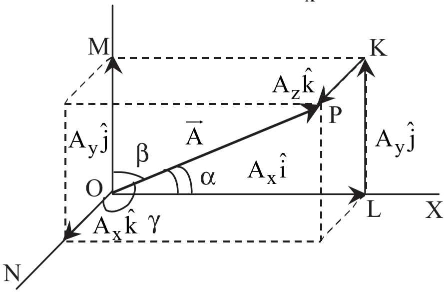


$$\vec{A}_1 = A_x\hat{i},\quad \vec{A}_2 = A_y\hat{j} \quad \Rightarrow \quad \vec{A} = A_x\hat{i} + A_y\hat{j}$$

The quantities A_x and A_y are called x- and y-components of the vector $\vec{A}$. A_x is itself not a vector but $A_x\hat{i}$ is a vector and so is $A_y\hat{j}$.

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta$$

It is clear from above equation that **a component of a vector can be positive, negative or zero depending on the value of θ.**



A vector $\vec{A}$ can be specified in a plane by two ways :

(a) Its magnitude A and the direction θ it makes with the x-axis, or

(b) Its components A_x and A_y.

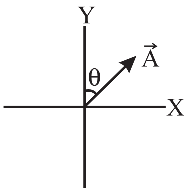
$$A = \sqrt{A_x^2 + A_y^2}, \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x}s$$

If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$ i.e. components of a vector perpendicular to itself is always zero.

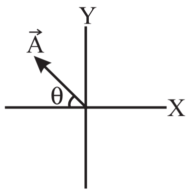
In three dimensions, a vector $\vec{A}$ in components along x, y and z-axis can be written as :

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP} \Rightarrow \vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$$

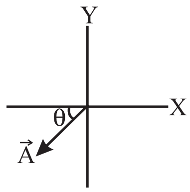
See following diagram carefully :



X Comp→ +A sinθ
Y Comp→ +A cosθ



X Comp→ –A sinθ
Y Comp→ +A cosθ



X Comp→ –A sinθ
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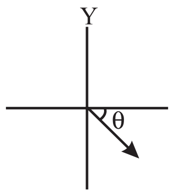


X Comp→ +A sinθ
Y Comp→ –A cosθ

ILLUSTRATION : 3

If $0.3\hat{i} + 0.4\hat{j} + c\hat{k}$ is a unit vector then find the value of c.

SOLUTION:

$$\sqrt{0.3^2 + 0.4^2 + c^2} = 1 \Rightarrow 0.09 + 0.16 + c^2 = 1 \Rightarrow c^2 = 1 - 0.25 = 0.75 \Rightarrow c = \sqrt{0.75}$$

ILLUSTRATION : 4

What is the magnitude of the vector (i) $3\hat{i} - 4\hat{j}$ and (ii) $2\hat{i} + 3\hat{j} - 6\hat{k}$?

SOLUTION:

We know that the magnitude of a vector $x\hat{i} + y\hat{j} + z\hat{k}$ is given by $\sqrt{x^2 + y^2 + z^2}$ therefore,

(i) $|3\hat{i} - 4\hat{j}| = \sqrt{(3)^2 + (-4)^2 + (0)^2} = 5$ units

(ii) $|2\hat{i} + 3\hat{j} - 6\hat{k}| = \sqrt{(2)^2 + (3)^2 + (-6)^2} = 7$ units

ILLUSTRATION : 5

Two forces $\vec{A}$ and $\vec{B}$ have resultant $\vec{R}_1$. If $\vec{B}$ is doubled, the new resultant $\vec{R}_2$ is at right angle to $\vec{A}$. Find value of R_1 and R_2 in terms of A and B.

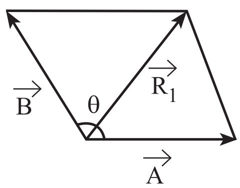
SOLUTION:

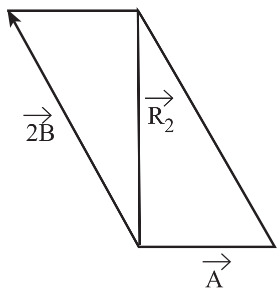
Let angle between $\vec{A}$ and $\vec{B}$ be θ, so angle between $\vec{A}$ and $2\vec{B}$ is also θ.

$\Rightarrow \quad R_1^2 = A^2 + B^2 + 2AB\cos\theta$ .... (1)

and $\tan 90° = \dfrac{2B\sin\theta}{A + 2B\cos\theta} \to \infty$

$\Rightarrow \quad A + 2B\cos\theta = 0 \Rightarrow \cos\theta = -\dfrac{A}{2B}$ .... (2)





Substituting value of cos θ in eq. (1), we get, $R_1 = B$

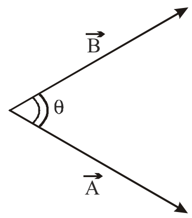
Also, $R_2^2 + A^2 = (2B)^2 = 4B^2 \Rightarrow R_2 = \sqrt{4B^2 - A^2}$

MULTIPLICATION OF VECTORS-SCALAR OR DOT PRODUCT

Scalar or Dot Product

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}.\vec{B}$ (read as $\vec{A}$ dot $\vec{B}$) is defined as *the product of their magnitude with cosine of angle between them.* Thus,

$\vec{A}.\vec{B} = AB\cos\theta$ (here θ is the angle between the vectors)



Properties of Scalar or Dot Product

(a) It is always a scalar.

(b) It is commutative, i.e., $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

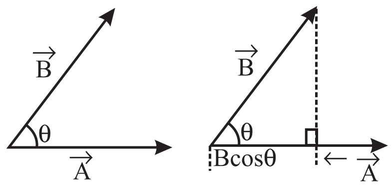
(c) It is distributive, i.e. $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

(d) As $\vec{A}.\vec{B} = AB\cos\theta$ therefore,

Angle between the vectors A and B, $\theta = \cos^{-1}\left[\frac{\vec{A}.\vec{B}}{AB}\right]$

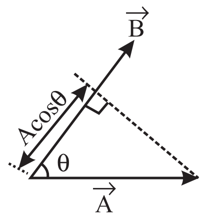
(e) $\vec{A}.\vec{B} = A\,(B\cos\theta) = B\,(A\cos\theta)$

Geometrically, B cos θ is the projection of $\vec{B}$ onto $\vec{A}$ and A cos θ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A}.\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.



Component of $\vec{B}$ along $\vec{A}$ = $B\cos\theta = \frac{\vec{A}.\vec{B}}{A} = \hat{A}.\vec{B}$

Component of $\vec{A}$ along $\vec{B}$ = $A\cos\theta = \frac{\vec{A}.\vec{B}}{B} = \vec{A}.\hat{B}$



(f) Scalar product of two vectors will be maximum when cos θ = max = 1 = 1, i.e. θ = 0°, i.e., vectors are parallel $\Rightarrow (\vec{A}.\vec{B})_{max} = AB$

(g) If the scalar product of two non-zero vectors vanishes then the vectors are orthogonal.

(h) The scalar product of a vector by itself is termed as self dot product and is given by

$$(\vec{A})^2 = \vec{A}.\vec{A} = AA\cos\theta = A^2 \Rightarrow A = \sqrt{\vec{A}.\vec{A}}$$

(i) In case of unit vector $\hat{n}$, $\hat{n}.\hat{n} = 1\times1\times\cos0° = 1$

$\Rightarrow \hat{\mathbf{n}}.\hat{\mathbf{n}} = \hat{\mathbf{i}}.\hat{\mathbf{i}} = \hat{\mathbf{j}}.\hat{\mathbf{j}} = \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1$

(j) In case of orthogonal unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$; $\hat{\mathbf{i}}.\hat{\mathbf{j}} = \hat{\mathbf{j}}.\hat{\mathbf{k}} = \hat{\mathbf{k}}.\hat{\mathbf{i}} = 0$

(k) Scalar product in cartesian coordinates

$\vec{A}.\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$

Example:

(i) Work (W) : In physics for constant force work is defined as, $W = Fs\cos\theta$ ... (1)

But by definition of scalar product of two vectors, $\vec{F}.\vec{s} = Fs\cos\theta$ ... (2)

So from eq. (1) and (2) $W = \vec{F}.\vec{s}$ i.e., work is the scalar product of force and displacement.

(ii) Power (P) : As $W = \vec{F}.\vec{s}$ or $\frac{dW}{dt} = \vec{F}.\frac{d\vec{s}}{dt}$ (as $\vec{F}$ is constant)

or $P = \vec{F}.\vec{v}$ i.e., power is the scalar product of force with velocity. [as $\frac{dW}{dt} = P$ and $\frac{d\vec{s}}{dt} = \vec{v}$]

Uses of Scalar Product

It is used to find

1. angle between two vectors.
2. condition for two vectors to be perpendicular to each other.
3. physical quantities based on scalar product.

ILLUSTRATION : 6

Prove that vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j}$ are perpendicular to each other.

SOLUTION:

Here, $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$, and $\vec{B} = 2\hat{i} - \hat{j}$

Two vectors are perpendicular to each other if, $\vec{A}.\vec{B} = 0$

Now $\vec{A}.\vec{B} = (\hat{i} + 2\hat{j} + 3\hat{k}) . (2\hat{i} - \hat{j})$

$= 1 \times 2 + 2 \times (-1) + 3 \times (0) = 2 - 2 + 0 = 0$

Since $\vec{A}.\vec{B} = 0$, so vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

ILLUSTRATION : 7

Find the angle between the vectors $\vec{A} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{B} = -\hat{i} + 2\hat{j} - \hat{k}$.

SOLUTION:

Here $\vec{A} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{B} = -\hat{i} + 2\hat{j} - \hat{k}$

We know that $\vec{A}.\vec{B} = AB\cos\theta$ or $\cos\theta = \dfrac{\vec{A}.\vec{B}}{AB}$

Now $A = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$, $B = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$

$\vec{A}.\vec{B} = (\hat{i} + \hat{j} - 2\hat{k}).(-\hat{i} + 2\hat{j} - \hat{k}) = -1 + 2 + 2 = 3$

$$\therefore \quad \cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{3}{\sqrt{6} \times \sqrt{6}} = \frac{3}{6} = \frac{1}{2} \quad \Rightarrow \quad \theta = 60°$$

MULTIPLICATION OF VECTORS-VECTOR OR CROSS PRODUCT

Vector or Cross Product

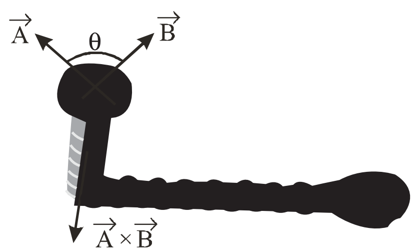
The vector product or cross product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them, and direction perpendicular to the plane containing the two vectors in accordance with right hand screw rule. $\vec{C} = \vec{A} \times \vec{B}$

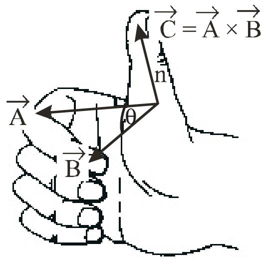
Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product written as $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A} \times \vec{B} = AB\sin\theta\hat{n}$$

Direction of Vector or Cross Product

The direction of $\vec{A} \times \vec{B}$, i.e. $\vec{C}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ in the sense of advance of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the small angle between them.





Thus, if a right handed screw whose axis is perpendicular to the plane framed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A}\times\vec{B}$ i.e. $\vec{C}$.

$C \perp A, C \perp B$ Hence, $\vec{C}.\vec{A} = 0,\ \vec{C}.\vec{B} = 0$

Properties of Vector or Cross Product

(i) Vector product of any two vectors is always a vector perpendicular to the plane containing these two vectors, i.e., orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

(ii) Vector product of two vectors is not *commutative*, i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ [but = $-\ \vec{B}\times\vec{A}$]

Here it is worthnoting that $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$, i.e. in case of vector $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ magnitudes are equal but directions are opposite.

(iii) The vector product is *distributive* when the order of the vectors is strictly maintained, i.e. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

(iv) The vector product of two vectors will be maximum when $\sin\theta = \max = 1$, i.e., $\theta = 90°$

$[\vec{A}\times\vec{B}]_{max} = AB\sin\theta\ \hat{n}$, i.e., vector product is maximum if the vectors are orthogonal.

(v) The vector product of two non-zero vectors will be minimum when

$$|\sin\theta| = \text{minimum} = 0, \text{ i.e., } \theta = 0° \text{ or } 180° \quad [\vec{A}\times\vec{B}]_{min} = 0$$

i.e., if the vector product of two non-zero vectors vanishes, the vectors are collinear.

(vi) The self cross product, i.e. product of a vector by itself vanishes, i.e., null vector, $\vec{A}\times\vec{A} = AA\sin 0°\ \hat{n} = \vec{0}$.

(vii) In case of unit vector $\hat{n}\times\hat{n} = \vec{0}$ so that $\mathbf{\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}}$

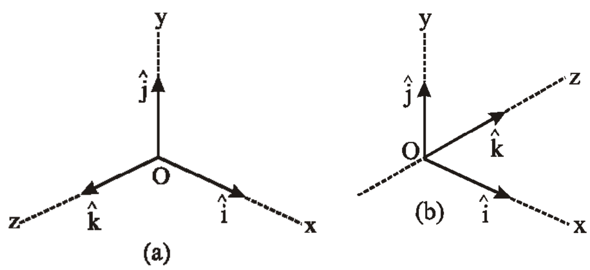
(viii) Vector or cross product of orthogonal unit vectors.

(a) A right-handed coordinate system, in which

$\mathbf{\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i}}$ and $\mathbf{\hat{k}\times\hat{i} = \hat{j}}$.

(b) A left-handed coordinate system, in which

$\hat{i}\times\hat{j} = -\hat{k}$, and so on. We'll use only right-handed systems.



(ix) Vector or cross product in cartesian coordinates

$$A\times B = (A_x\hat{i} + A_y\hat{J} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{J} + B_z\hat{k})$$

$$= (A_zB_z - A_zB_y)\hat{i} - (A_xB_z - B_xA_z)\hat{J} + (A_xB_y - A_yB_x)\hat{k}$$

Since vector product of two vectors is a vector, vector physical quantities (particularly representing rotational effects) like torque, angular momentum, velocity and force on a moving charge in a magnetic field can be expressed as the vector product of two vectors. It is well-established in physics.

Example:

(i) Torque $\vec{\tau} = \vec{r}\times\vec{F}$

(ii) Angular momentum $\vec{L} = \vec{r}\times\vec{p}$

(iii) Linear velocity $\vec{v} = \vec{\omega}\times\vec{r}$

(iv) Force on a charged particle q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q\,(\vec{v}\times\vec{B})$

(v) Torque on a dipole in electric and magnetic field, $\vec{\tau}_E = \vec{p}\times\vec{E}$ and $\vec{\tau}_B = \vec{M}\times\vec{B}$.

(vi) Force on current carrying conductor in magnetic field $\vec{F} = i\,(\vec{\ell}\times\vec{B})$.

Uses of Vector or Cross Product

Vector product or cross product is used to find

1. condition for two vectors to be parallel to each other $\vec{A}\times\vec{B}=0$
2. physical quantities based on cross product.
3. area of parallelogram/triangle.
   (i) The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|a\times\vec{b}|$.
   (ii) The area of a parallelogram or a plane quadrilateral with diagonals $\vec{d}_1$ and $\vec{d}_2$ is $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$.
   (iii) The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|a\times\vec{b}|$.

(iv) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are position vectors of vertices of a Δ ABC, then its area = $\frac{1}{2}|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})|$

(v) Three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear, if $(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})=0$

LLUSTRATION: 8

Considering two vectors, $F=(4\vec{i}-10\vec{j})$ N and $\vec{r}=(-5\vec{i}-3\vec{j})$ m. Compute $\vec{r}\times\vec{F}$ and state what physical quantity it represents ?

SOLUTION:

As $\vec{F}=(4\vec{i}-10\vec{j}+0\vec{k})$ and $r=(-5\vec{i}-3\vec{j}+0\vec{k})$

$$\vec{r}\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ F_x & F_y & F_z\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -5 & -3 & 0\\ 4 & -10 & 0\end{vmatrix} \quad \text{i.e.,} \quad \vec{r}\times\vec{F}=\vec{k}\,(50+12)=62\,\vec{k}\text{ N m}$$

The given physical quantity $\vec{r}\times\vec{F}$ represents torque (i.e., moment of force) if $\vec{F}$ represents force and $\vec{r}$ is the position vector.

ILLUSTRATION: 9

Find the area of the triangle formed by the tips of the vectors $\vec{a}=\hat{i}-\hat{j}-3\hat{k}$, $\vec{b}=4\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

SOLUTION:

Let *ABC* is the triangle formed by the tips of the given vectors, then

$$\overrightarrow{AB}=\vec{b}-\vec{a}$$
$$=(4\hat{i}-3\hat{j}+\hat{k})-(\hat{i}-\hat{j}-3\hat{k})=3\hat{i}-2\hat{j}+4\hat{k}$$

And $\overrightarrow{AC}=\vec{c}-\vec{a}$

$$=(3\hat{i}-\hat{j}+2\hat{k})-(\hat{i}-\hat{j}-3\hat{k})=2\hat{i}+5\hat{k}$$

Now $\overrightarrow{AB}\times\overrightarrow{AC}=(3\hat{i}-2\hat{j}+4\hat{k})\times(2\hat{i}+5\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -2 & 4\\ 2 & 0 & 5\end{vmatrix}=\hat{i}(-10-0)+\hat{j}(8-15)+\hat{k}(0+4)=-10\hat{i}-7\hat{j}+4\hat{k}$$

and $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(-10)^2+(-7)^2+(4)^2}=\sqrt{165}=12.8$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\times 12.8=6.4\text{ m}^2$