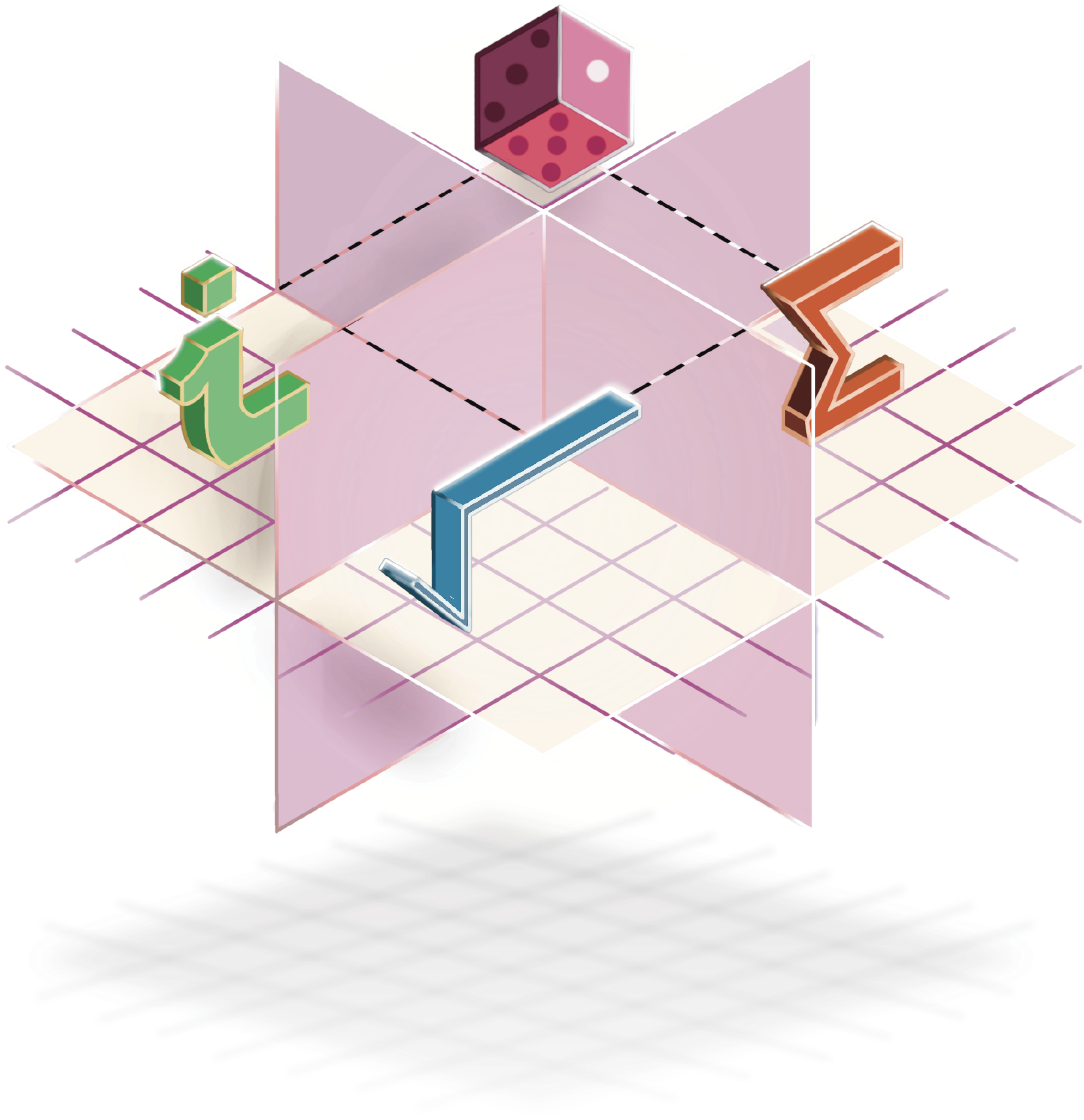
Mathematics

Class 11



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BINOMIAL THEOREM

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1. BINOMIAL THEOREM

If a, b $\in$ R and n $\in$ N, then

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_n\, a^0 b^n$

REMARKS :

1. If the index of the binomial is n then the expansion contains n + 1 terms.
2. In each term, the sum of indices of a and b is always n.
3. Coefficients of the terms in binomial expansion equidistant from both the ends are equal.
4. $(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^{2-} ... + (-1)^n\, {}^nC_0 a^0 b^n$.

2. GENERAL TERM AND MIDDLE TERMS IN EXPANSION OF (a + b)^n

$t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

t_{r+1} is called a general term for all r $\in$ W and $0 \le r \le n$. Using this formula we can find any term of the expansion.

MIDDLE TERM (S) :

1. In $(a + b)^n$ if n is even then the number of terms in the expansion is odd. Therefore there is only one middle term and it is $\left(\frac{n+2}{2}\right)^{th}$ term.
2. In $(a + b)^n$, if n is odd then the number of terms in the expansion is even. Therefore there are two middle terms and those are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

3. NUMERICALLY GREATEST TERM

The term with greatest numerical/absolute value in the expansion of $(a + b)^n$ can be found in following way

(i) Find the value of $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$

(ii) If it equals an integer, say m, then t_m and t_{m+1} are numerically greatest terms.

(iii) If it is not an integer, then t_{m+1} is numerically greatest term (where m is the integral part of $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$).

Also middle terms in binomial expansions have the greatest binomial coefficients. (${}^nC_0, {}^nC_1, {}^nC_2 .... {}^nC_n$ are called Binomial Coefficients).

4. BINOMIAL COEFFICIENTS

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ in the expansion of $(a+b)^n$ are called the binomial coefficients and denoted by $C_0, C_1, C_2, ..., C_n$ respectively

Now

$(1 + x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + ... + {}^nC_n x^n$ ..... (i)

Put x = 1.

$(1 + 1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore\ 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore\ {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

$\therefore\ C_0 + C_1 + C_2 + ... + C_n = 2^n$

$\therefore$ The sum of all binomial coefficients is 2^n.

Put x = –1, in equation (i),

$(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\, {}^nC_n$

$\therefore\ 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\, {}^nC_n$

$\therefore\ {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\, {}^nC_n = 0$

$\therefore$ ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ...$

$\therefore$ $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...$

$C_0, C_2, C_4, ...$ are called as even coefficients

$C_1, C_3, C_5 ...$ are called as odd coefficients

Let $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = k$

Now $C_0 + C_1 + C_2 + C_3 + ... + C_n = 2^n$

$\therefore$ $(C_0 + C_2 + C_4 + ...) + (C_1 + C_3 + C_5 ...) = 2^n$

$\therefore$ $k + k = 2^n$

$2k = 2^n$

$\therefore$ $k = \dfrac{2^n}{2}$

$\therefore$ $k = 2^{n-1}$

$\therefore$ $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

$\therefore$ The sum of even coefficients = The sum of odd coefficients $= 2^{n-1}$

Properties of Binomial Coefficient

(i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(ii) $C_0 - C_1 + C_2 - .... + (-1)^n C_n = 0$

(iii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$.

(iv) ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2 \text{ or } r_1 + r_2 = n$

(v) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(vi) $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$

Some Important Results

(i) Differentiating $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, on both sides we have,

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_nx^{n-1}$ ....(1)

Put $x = 1$

$\Rightarrow$ $n2^{n-1} = C_1 + 2C_2 + 3C_3 + ... + nC_n$

Put $x = -1$

$\Rightarrow$ $0 = C_1 - 2C_2 + ... + (-1)^{n-1} nC_n$.

Differentiating (1) again and again we will have different results.

(ii) Integrating $(1+x)^n$, we have,

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$$

(where C is a constant)

Put $x = 0$, we get $C = -\dfrac{1}{(n+1)}$

Therefore

$$\frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1} \quad ...(2)$$

Put $x = 1$ in (2) we get

$$\frac{2^{n+1} - 1}{n+1} = C_0 + \frac{C_1}{2} + ... + \frac{C_n}{n+1}$$

Put $x = -1$ in (2) we get,

$$\frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ....$$

5. BINOMIAL THEOREM FOR ANY INDEX

If n is any real number and $|x| < 1$ then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

Here there are infinite number of terms in the expansion, The general term is given by

$$t_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)x^r}{r!}, r \geq 0$$

NOTES :

(i) Expansion is valid only when $-1 < x < 1$

(ii) ${}^nC_r$ can not be used because it is defined only for natural number, so ${}^nC_r$ will be written as

$$\frac{n(n-1).........(n-r+1)}{r!}$$

(iii) As the series never terminates, the number of terms in the series is infinite.

(iv) If first term is not 1, then make it unity in the following way. $(a+x)^n = a^n(1+x/a)^n$ if $\left|\dfrac{x}{a}\right| < 1$

NOTES :

While expanding $(a + b)^n$ where n is a negative integer or a fraction, reduce the binomial to the form in which the first term is unity and the second term is numerically less than unity.

Particular expansion of the binomials for negative index, $|x| < 1$

1. $\frac{1}{1+x} = (1+x)^{-1}$

   $= 1 - x + x^2 - x^3 + x^4 - x^5 + ...$

2. $\frac{1}{1-x} = (1-x)^{-1}$

   $= 1 + x + x^2 + x^3 + x^4 + x^5 + ...$

3. $\frac{1}{(1+x)^2} = (1+x)^{-2}$

   $= 1 - 2x + 3x^2 - 4x^3 + ...$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2}$

   $= 1 + 2x + 3x^2 + 4x^3 + ...$

6. MULTINOMIAL EXPANSION

In the expansion of $(x_1 + x_2 + ... + x_n)^m$ where $m, n \in N$ and $x_1, x_2, ..., x_n$ are independent variables, we have

(i) Total number of terms $= {}^{m+n-1}C_{n-1}$

(ii) Coefficient of $x_1^{r_1} x_2^{r_2} x_3^{r_3} ....x_n^{r_n}$ (where $r_1 + r_2 + ... + r_n = m, r_i \in N \cup \{0\}$ is $\frac{m!}{r_1!r_2!.......r_n!}$

(iii) Sum of all the coefficients is obtained by putting all the variables x_i equal to 1.

SOLVED EXAMPLES

Example – 1

Expand

(i) $(2x^2+3)^4$ (ii) $\left(2x^2-\frac{1}{x}\right)^6$

Sol. **(i)** $(2x^2+3)^4=$

$={}^4C_0(2x^2)^4(3)^0+{}^4C_1(2x^2)^3(3)^1+{}^4C_2(2x^2)^2$

$(3)^2+{}^4C_3(2x^2)^1(3)^3+{}^4C_4(2x^2)^0(3)^4$

$=(1)\,16x^8(1)+4\,(8x^6)\,(3)+6\,(4x^4)\,(9)+4\,(2x^2)\,27+(1)(1)\,81$

$$\because \left\{\begin{array}{l} {}^4C_0={}^4C_4=1,\ {}^4C_1={}^4C_3=4 \\ {}^4C_2=\dfrac{4!}{2!\,2!}=\dfrac{4\times3\times2!}{2!\times2}=6 \end{array}\right\}$$

$=16x^8+96x^6+216x^4+216x^2+81$

(ii) $\left(2x^2-\frac{1}{x}\right)^6={}^6C_0\,(2x^2)^6\left(\frac{1}{x}\right)^0-$

${}^6C_1(2x^2)^5\left(\frac{1}{x}\right)^1+{}^6C_2\,(2x^2)^4\left(\frac{1}{x}\right)^2-$

${}^6C_3\,(2x^2)^3\left(\frac{1}{x}\right)^3+{}^6C_4\,(2x^2)^2$

$\left(\frac{1}{x}\right)^4-{}^6C_5\,(2x^2)\left(\frac{1}{x}\right)^5+{}^6C_6\,(2x^2)^0\left(\frac{1}{x}\right)^6$

$=(1)\,64x^{12}(1)-(6)\,(32)\,x^{10}\times\frac{1}{x}+15\,(16)\,x^8\times\frac{1}{x^2}$

$-20\times8x^6\times\frac{1}{x^3}+15(4)\,x^4\times\frac{1}{x^4}$

$-6(2x^2)\times\frac{1}{x^5}+(1)\,(1)\frac{1}{x^6}$

$$\because \left\{\begin{array}{l} {}^6C_0={}^6C_6=1,\ {}^6C_1={}^6C_5=6 \\ {}^6C_2=\dfrac{6!}{2!4!}=\dfrac{6\times5\times4!}{2\times4!}=15 \\ {}^6C_3=\dfrac{6!}{3!3!}=\dfrac{6\times5\times4\times3!}{3\times2\times3!}=20 \end{array}\right\}$$

$=64x^{12}-192x^9+240x^6-160x^3+60-\frac{12}{x^3}+\frac{1}{x^6}$

Example – 2

Expand $(1+x+x^2)^3$.

Sol. Let $y=x+x^2$. Then,

$(1+x+x^2)^3=(1+y)^3={}^3C_0+{}^3C_1\,y+{}^3C_2y^2+{}^3C_3y^3$

$=1+3y+3y^2+y^3=1+3\,(x+x^2)+3\,(x+x^2)^2+(x+x^2)^3$

$=1+3\,(x+x^2)+3(x^2+2x^3+x^4)+\{{}^3C_0x^3\,(x^2)^0+{}^3C_1\,x^{3-1}\,(x^2)^1$

$+{}^3C_2x^{3-2}\,(x^2)^2+{}^3C_3\,x^0\,(x^2)^3\}$

$=1+3\,(x+x^2)+3(x^2+2x^3+x^4)+(x^3+3x^4+3x^5+x^6)$

$=x^6+3x^5+6x^4+7x^3+6x^2+3x+1$

Example – 3

Prove that $(\sqrt{5}+1)^5-(\sqrt{5}-1)^5=352$

Sol. $(\sqrt{5}+1)^5-(\sqrt{5}-1)^5=$

$$=\left[\begin{array}{c} {}^5C_0(\sqrt{5})^5+{}^5C_1(\sqrt{5})^4(1)+{}^5C_2(\sqrt{5})^3(1)^2 \\ +{}^5C_3(\sqrt{5})^2(1)^3+{}^5C_4(\sqrt{5})(1)^4 \\ +{}^5C_5(\sqrt{5})^0(1)^5 \end{array}\right]$$

$$-\left[\begin{array}{c} {}^5C_0(\sqrt{5})^5-{}^5C_1(\sqrt{5})^4(1)+{}^5C_2(\sqrt{5})^3(1)^2 \\ -{}^5C_3(\sqrt{5})^2(1)^3+{}^5C_4(\sqrt{5})(1)^4 \\ -{}^5C_5(\sqrt{5})^0(1)^5 \end{array}\right]$$

$$=2\left[{}^5C_1 5^2+{}^5C_3.5+{}^5C_5\right]$$

$$=2[5\times25+10\times5+1]$$

$$\left\{\begin{array}{l} {}^5C_0={}^5C_5=1;\ {}^5C_4=5; \\ {}^5C_2={}^5C_3=\frac{5.4}{2.1}=10;\ {}^5C_1=5 \end{array}\right\}$$

$=2[125+51]$

$=352$

Example – 4

Using binomial theorem compute $(99)^5$.

Sol. $(99)^5=(100-1)^5={}^5C_0(100)^5-{}^5C_1(100)^4+{}^5C_2(100)^3$

$-{}^5C_3(100)^2+{}^5C_4(100)^1-{}^5C_5(100)^0$

$=(100)^5-5\times(100)^4+10\times(100)^3-10\times(100)^2+5\times100-1$

$=10^{10}-5\times10^8+10^7-10^5+5\times10^2-1$

$=(10^{10}+10^7+5\times10^2)-(5\times10^8+10^5+1)$

$=10010000500-500100001=9509900499$

Example – 5

Use the binomial theorem to find the exact value of $(10.1)^5$.

Sol. $(10.1)^5=(10+0.1)^5$

$=10^5+{}^5C_1 10^4(.1)+{}^5C_2 10^3(.1)^2+{}^5C_3 10^2(.1)^3$

$+{}^5C_4 10(.1)^4+{}^5C_5(.1)^5$

$=100000+5\times10^4(.1)+10\times(10^3)(.01)+10\times10^2(.001)$

$+5\times10(.0001)+0.00001$

$=100000+5000+100+1+0.005+0.00001=105101.00501$

Example – 6

Prove that $\sum_{r=1}^{5}{}^5C_r=31$

Sol. $\sum_{r=1}^{5}{}^5C_r={}^5C_1+{}^5C_2+{}^5C_3+{}^5C_4+{}^5C_5$

$$=\frac{5!}{1!4!}+\frac{5!}{2!3!}+\frac{5!}{3!2!}+\frac{5!}{4!1!}+\frac{5!}{5!0!}$$

$$=\frac{5}{1}+\frac{5.4}{2}+\frac{5.4}{2}+\frac{5}{1}+1$$

$=5+10+10+5+1=31$

Example – 7

Find n, if ${}^nC_6:{}^{n-3}C_3=33:4$.

Sol. Given, ${}^nC_6:{}^{n-3}C_3=33:4$.

$$\therefore\ \frac{n!}{6!(n-6)!}\times\frac{3!(n-3-3)!}{(n-3)!}=\frac{33}{4}$$

$$\text{or}\ \frac{n!}{(n-3)!}.\frac{3!}{6!}=\frac{33}{4}\ \text{or}\ \frac{n(n-1)(n-2)}{6.5.4}=\frac{33}{4}$$

or $n(n-1)(n-2)=6.5.33=11.3.3.2.5$

or $n(n-1)(n-2)=11.(3.3).(2.5)=11.10.9$ $\therefore$ $n=11$

Example – 8

If ${}^nC_8={}^nC_6$ determine n and hence ${}^nC_2$.

Sol. Given, ${}^nC_8={}^nC_6$

We know that ${}^nC_x={}^nC_y$ then $x=y$ or $x+y=n$

$\Rightarrow n=8+6$

$\Rightarrow$ $n=14$

Now ${}^nC_2={}^{14}C_2=\frac{14\times13}{2!}=91$

Example – 9

If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.

Sol. We know that if ${}^{n}C_{x} = {}^{n}C_{y}$, then x = y or x + y = n

$${}^{15}C_{3r} = {}^{15}C_{r+3}$$

$\therefore$ Either $3r = r + 3 \Rightarrow r = \frac{3}{2}$,

which is not possible, since r is an integer.

or $3r + r + 3 = 15 \Rightarrow r = 3$.

Hence r = 3.

Example – 10

Find the third term in the expansion of $\left(2x^2 + \frac{3}{2x}\right)^8$

Sol. Let $a = 2x^2$, $b = \frac{3}{2x}, n = 8$

For third term, r = 2

$t_{r+1} = {}^{n}C_r a^{n-r} b^r$

$$= {}^{8}C_2 (2x^2)^{8-2}\left(\frac{3}{2x}\right)^2$$

$$= \frac{8.7.6!}{2!6!}(2x^2)^6 \times \frac{9}{4x^2}\left[\because {}^{8}C_2 = \frac{8!}{2!6!}\right]$$

$$= \frac{8.7}{2} \times 2^6 \times x^{12} \times \frac{9}{4x^2}$$

$$= 63 \times 64x^{10} = 4032x^{10}$$

Example – 11

Find the fifth term in the expansion of $\left(x^2 - \frac{4}{x^3}\right)^{11}$

Sol. Let, $a = x^2$, $b = \frac{-4}{x^3}, n = 11$

For fifth term, r = 4

$\therefore \quad t_{r+1} = {}^{n}C_r a^{n-r} . b^r$

$\therefore \quad t_5 = {}^{11}C_4 (x^2)^{11-4}\left(\frac{-4}{x^3}\right)^4$

$\therefore \quad t_5 = \frac{11!}{4!\,7!}(x^2)^7 \times \frac{4^4}{x^{12}}$

$\therefore \quad t_5 = \frac{11 \times 10 \times 9 \times 8 \times 7!}{4 \times 3 \times 2 \times 7!} x^{14} \times \frac{256}{x^{12}}$

$\therefore \quad t_5 = 330 \times 256x^2 \quad \Rightarrow t_5 = 84480x^2$

Example – 12

Given that the 4th term in the expansion of $\left(px + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, find n and p.

Sol. Given expansion is $\left(px + \frac{1}{x}\right)^n$

Given, $T_4 = \frac{5}{2}$

$\therefore \quad {}^{n}C_3 (px)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2}$

$\Rightarrow \quad {}^{n}C_3 p^{n-3} x^{n-3} . \frac{1}{x^3} = \frac{5}{2}$

$\Rightarrow \quad \frac{n!}{3!(n-3)!}.p^{n-3} x^{n-6} = \frac{5}{2}$ ...(1)

Since R.H.S. of (1) is independent of x,

therefore n – 6 = 0 $\quad \therefore \quad$ n = 6.

From (1), $\quad \frac{6!}{3!\,3!}.p^3 = \frac{5}{2}$

$\Rightarrow \quad 20p^3 = \frac{5}{2}$

$\Rightarrow \quad p^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \quad \therefore \; p = \frac{1}{2}$.

Hence n = 6 and $p = \frac{1}{2}$.

Example – 13

Given positive integers $r > 1$, $n > 2$ and the coefficient of $(3r)^{th}$ and $(r+2)^{th}$ terms in the binomial expansion of $(1+x)^{2n}$ are equal . Then :

(a) $n = 2r$ (b) $n = 2r + 1$

(c) $n = 3r$ (d) none of these

Ans. (a)

Sol. In the expansion $(1+x)^{2n}$, $t_{3r} = {}^{2n}C_{3r-1}\,(x)^{3r-1}$

and $t_{r+2} = {}^{2n}C_{r+1}\,(x)^{r+1}$

Since, binomial coefficients of t_{3r} and t_{r+2} are equal.

$\therefore\ {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow 3r - 1 = r + 1$ or $2n = (3r-1) + (r+1)$

$\Rightarrow 2r = 2$ or $2n = 4r$

$\Rightarrow r = 1$ or $n = 2r$

But $r > 1$

$\therefore$ We take, $n = 2r$

Example – 14

If the coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1 : 7 : 42, find n.

Sol. Let the three consecutive terms in the expansion of $(1+a)^n$ be rth, $(r+1)$th and $(r+2)$th terms respectively.

In the expansion of $(1+a)^n$,

coefficient of rth term $= {}^nC_{r-1}$,

coefficient of $(r+1)$th term $= {}^nC_r$.

coefficient or $(r+2)$th term $= {}^nC_{r+1}$

Given, ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42$.

$$\therefore\ \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7}$$

$$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!}\cdot\frac{r!(n-r)!}{n!} = \frac{1}{7}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{1}{7}$$

$\Rightarrow 7r = n - r + 1$

$\Rightarrow n - 8r = -1$ .... (1)

$$\text{And } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{7}{42}$$

$$\Rightarrow \frac{n!}{r!(n-r)!}\cdot\frac{(r+1)!(n-r-1)!}{n!} = \frac{1}{6}$$

$$\Rightarrow \frac{r+1}{n-r} = \frac{1}{6}$$

$\Rightarrow 6r + 6 = n - r \quad \therefore\ n - 7r = 6$ .... (2)

Now, (2) – (1) $\Rightarrow r = 7$

From (1), $n = 55$.

Example – 15

If in the expansion of $(1+x)^n$, the coefficients of 14th, 15th and 16th terms in A.P. find n.

Sol. The coefficients of 14th, 15th and 16th tems in the expansion of $(1+x)^n$ will be ${}^nC_{13}$, ${}^nC_{14}$ and ${}^nC_{15}$ respectively.

Given, ${}^nC_{13}$, ${}^nC_{14}$ and ${}^nC_{15}$ are in A.P.

$\therefore\ {}^nC_{14} - {}^nC_{13} = {}^nC_{15} - {}^nC_{14}$

or $2\,.\,{}^nC_{14} = {}^nC_{13} + {}^nC_{15}$

$$\text{or}\quad 2.\frac{n!}{(14)!(n-14)!} = \frac{n!}{(13)!(n-13)!} + \frac{n!}{(15)!(n-15)!}$$

Multiplying both sides by $15!\,(n-13)!$, we get

$$2.\frac{15!(n-13)!}{14!(n-14)!} = \frac{15!(n-13)!}{13!(n-13)!} + \frac{15!(n-13)!}{15!(n-15)!}$$

or $2.15\,(n-13) = 15.14 + (n-13)(n-14)$

or $30n - 390 = 210 + n^2 - 27n + 182$

or $n^2 - 57n + 782 = 0$

or $(n-34)(n-23) = 0$

Hence $n = 23$ or 34.

Example – 16

The term independent of x in $\left[\sqrt{\left(\frac{x}{3}\right)}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is:

(a) 1 (b) $^{10}C_1$

(c) 5/12 (d) none of these

Ans. (d)

Sol. $T_{r+1} = {}^{10}C_r\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\sqrt{\frac{3}{2x^2}}\right)^r$

Equating x power to zero

$$\frac{10-r}{2}-r=0$$

$$10-3r=0$$

$$\Rightarrow r=\frac{10}{3}$$

Independent of 'x' term is not possible

Example – 17

Find the constant term (term independent of x) in the expansion of

(i) $\left(2x+\frac{1}{3x^2}\right)^9$ (ii) $\left(x-\frac{2}{x^2}\right)^{15}$

Sol. Let $a=2x,\ b=\frac{1}{3x^2}, n=9$

$$t_{r+1} = {}^nC_r\, a^{n-r}\,.\, b^r$$

$$t_{r+1} = {}^9C_r(2x)^{9-r}\left(\frac{1}{3x^2}\right)^r$$

$$t_{r+1} = {}^9C_r(2)^{9-r}\left(\frac{1}{3}\right)^r x^{-2r}.x^{9-r}$$

$$t_{r+1} = {}^9C_r(2)^{9-r}\left(\frac{1}{3}\right)^r x^{9-3r}$$

To get term independent of x, must have

$x^{9-3r}=x^0$

$9-3r=0 \quad \Rightarrow -3r=-9 \quad \Rightarrow r=3$

$$\therefore\ {}^9C_3(2)^{9-3}\left(\frac{1}{3}\right)^3$$

$$\frac{9!}{3!6!}\times 2^6\times\frac{1}{27}=\frac{9\times8\times7\times6!}{3\times2\times6!}\times64\times\frac{1}{27}$$

$$=\frac{28\times64}{9}=\frac{1792}{9}$$

Constant term independent of $x=\frac{1792}{9}$

(ii) Let $a=x,\ b=\frac{-2}{x^2}, n=15$

$$t_{r+1} = {}^nC_r\, a^{n-r}\,.\, b^r$$

$$t_{r+1} = {}^{15}C_r(x)^{15-r}\left(\frac{-2}{x^2}\right)^r$$

$$t_{r+1} = {}^{15}C_r(x)^{15-r}(-2)^r x^{-2r}$$

$$t_{r+1} = {}^{15}C_r(-2)^r(x)^{15-3r}$$

To get constant term independent of x,

$x^{15-3r}=x^0$

$15-3r=0 \quad \Rightarrow -3r=-15 \quad \Rightarrow r=5$

$$\therefore\ {}^{15}C_5(-2)^5=\frac{15!}{5!10!}(-32)$$

$$=\frac{15\times14\times13\times12\times11\times10!}{5\times4\times3\times2\times10!}\times-32$$

$=-77\times39\times32=-96096$

Constant term independent of $x=-96096$

Example – 18

Find the middle term (s) in the expansion of

(i) $\left(\frac{x}{y}+\frac{y}{x}\right)^{12}$ (ii) $\left(x^4-\frac{1}{x^3}\right)^{11}$

Sol. **(i)** Let $a=\frac{x}{y}, b=\frac{y}{x}, n=12$

n is even.

$\therefore \left(\frac{n+2}{2}\right)=\left(\frac{12+2}{2}\right)=\frac{14}{2}=7$

7^{th} term is middle term,

$t_{r+1} = {}^nC_r\, a^{n-r}\,.\, b^r$

For 7th term, r = 6

$$t_7 = {}^{12}C_6\left(\frac{x}{y}\right)^{12-6}\left(\frac{y}{x}\right)^6$$

$$t_7 = \frac{12!}{6!6!}\times\left(\frac{x}{y}\right)^6\times\left(\frac{y}{x}\right)^6$$

$$t_7 = \frac{12\times11\times10\times9\times8\times7\times6!}{6\times5\times4\times3\times2\times6!}$$

$t_7 = 77\times12 = 924$

$\therefore$ Middle term = 924

(ii) Let $a = x^4,\ b=\frac{-1}{x^3}, n=11$

n is odd. $\left(\frac{n+1}{2}\right)=\left(\frac{11+1}{2}\right)=6,$

$\left(\frac{n+3}{2}\right)=\left(\frac{11+3}{2}\right)=7$

6th and 7th term are middle term,

$t_{r+1} = {}^nC_r a^{n-r}\,.\, b^r$

For t_6, r = 5

$$t_6 = {}^{11}C_5\,(x^4)^{11-5}\left(\frac{-1}{x^3}\right)^5$$

$$t_6 = \frac{11!}{5!\,6!}x^{24}\left(\frac{-1}{x^{15}}\right)$$

$$t_6 = \frac{11\times10\times9\times8\times7\times6!}{5\times4\times3\times2\times1\times6!}(-x^9)$$

$= -462\,x^9$

For t_7, r = 6

$$t_7 = {}^{11}C_6\,(x^4)^{11-6}\left(\frac{-1}{x^3}\right)^6$$

$$t_7 = \frac{11!}{6!5!}x^{20}\times\frac{1}{x^{18}}$$

$$t_7 = \frac{11\times10\times9\times8\times7\times6!}{6!\times5\times4\times3\times2}x^2$$

$t_7 = 11\times3\times2\times7 = 462x^2$

Example – 19

The middle term in expansion of $\left(x^2+\frac{1}{x^2}+2\right)^n$ is :

(a) $\frac{n!}{[(n/2)!]^2}$ (b) $\frac{2n!}{[(n/2)!]^2}$

(c) $\frac{1.3.5........(2n+1)}{n!}2^n$ (d) $\frac{(2n)!}{(n!)^2}$

Ans. (d)

Sol. Middle term in expansion of

$$\left(x^2+\frac{1}{x^2}+2\right)^n = \left(x+\frac{1}{x}\right)^{2n}$$

So, (n + 1)th term

$$\Rightarrow {}^{2n}C_n = \frac{(2n)!}{n!n!}$$

Example – 20

Find the middle terms(s) in the expansion of $\left(x^2+\frac{1}{x}\right)^7$

Sol. Let $a = x^2$, $b = \frac{1}{x}$, $n = 7$

n is odd.

$$\left(\frac{n+1}{2}\right)=\left(\frac{7+1}{2}\right)=4 \text{ and } \left(\frac{n+3}{2}\right)=\left(\frac{7+3}{2}\right)=5$$

4th and 5th terms are middle terms,

for t_4, $r = 3$

$t_{r+1} = {}^nC_r a^{n-r} . b^r$

$$t_4 = {}^7C_3 (x^2)^{7-3}\left(\frac{1}{x}\right)^3$$

$$t_4 = \frac{7!}{4!3!}\times(x^2)^4\times\frac{1}{x^3}$$

$$t_4 = \frac{7\times6\times5\times4!}{4!\times3\times2}\times x^8\times\frac{1}{x^3}$$

$t_4 = 35x^5$

For t_5, $r = 4$

$$t_5 = {}^7C_4\left(x^2\right)^{7-4}\left(\frac{1}{x}\right)^4$$

$$t_5 = \frac{7!}{4!3!}\times x^6\times\frac{1}{x^4}$$

$$t_5 = \frac{7\times6\times5\times4!}{4!\times3\times2}\times x^2 = 35x^2$$

Example – 21

Find the coefficient of x^9 in $(1 + 3x + 3x^2 + x^3)^{15}$.

Sol. $(1 + 3x + 3x^2 + x^3)^{15} = [(1 + x)^3]^{15} = (1 + x)^{45}$

$\therefore$ Coefficient of x^9 in $(1 + 3x + 3x^2 + x^3)^{15}$

= coefficient of x^9 in $(1 + x)^{45}$

$= {}^{45}C_9$ [Since in the expansion of $(1 + x)^n$, coefficient of $x^r = {}^nC_r$]

$$= \frac{45!}{9!36!}$$

Example – 22

Find the value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.

Sol. Given expression = ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.

$= {}^{47}C_4 + ({}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3)$

$= {}^{47}C_4 + ({}^{47}C_3 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3)$

$= ({}^{47}C_4 + {}^{47}C_3) + ({}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3)$

$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$ $[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$= ({}^{48}C_4 + {}^{48}C_3) + ({}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3)$

$= ({}^{49}C_4 + {}^{49}C_3) + ({}^{50}C_3 + {}^{51}C_3)$

$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3 = {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$.

Example – 23

The sum of coefficient in $(1 + x - 3x^2)^{2134}$ is

(a) –1 (b) 1

(c) 0 (d) 2^{2134}

Ans. (b)

Sol. Sum of co-efficients can be obtained by substituting x = 1

$\therefore$ Sum of co-efficients $= (1 + 1 - 3)^{2134} = (-1)^{2134} = 1$

Example – 24

The sum of the coefficients in the expansion of $(6a - 5b)^n$, where n is a positive integer, is

(a) 1 (b) –1

(c) 2^n (d) 2^{n-1}

Ans. (a)

Sol. Sum of coefficients we get when a = b = 1

$\Rightarrow (6 - 5)^n = 1$

Example – 25

Prove that $\sum_{r=0}^{n} 3^r \cdot {}^nC_r = 4^n$.

Sol. $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

$\therefore \quad {}^nC_0 x^0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n = (1+x)^n \qquad ....(1)$

Now $\sum_{r=0}^{n} 3^r \cdot {}^nC_r$

$= {}^nC_0 3^0 + {}^nC_1 3^1 + {}^nC_2 3^2 + ... + {}^nC_n 3^n$

$= (1+3)^n$

$= 4^n$.

Example – 26

Numerically greatest term, in the expansion of $(8 - 5x)^{18}$, (where $x = 2/5$) is :

(a) 1623×2^{24} (b) 1623×2^{22}

(c) 1623×2^{23} (d) none of these

Ans. (d)

Sol. $(8-5x)^{18} = 8^{18}\left(1-\frac{5x}{8}\right)^{18}$, $a = 1, b = -\frac{5}{8}\times\frac{2}{5} = -\frac{1}{4}$

$$\frac{(n+1)|b|}{|b|+|a|} = \frac{19\times\frac{1}{4}}{\frac{5}{4}} = 3.8$$

T_4 is greatest term

$T_4 = {}^{18}C_3\, 8^{15}\,(-2)^3$, so it is negative

Example – 27

Show that $2^{4n} - 2^n(7n+1)$ is some multiple of the square of 14, where n is a postive integer.

Sol. $2^{4n} - 2^n(7n+1) = (16)^n - 2^n(7n+1)$

$= (2+14)^n - 2^n \cdot 7n - 2^n$

$= (2^n + {}^nC_1 2^{n-1} \cdot 14 + {}^nC_2 2^{n-2} \cdot 14^2 + ... + 14^n) - 2^n \cdot 7n - 2^n$

$= 14^2({}^nC_2 2^{n-2} + {}^nC_3 2^{n-3} 14 + ... + 14^{n-2})$

$+ (2^n + {}^nC_1 \cdot 2^{n-1} \cdot 14 - 2^n \cdot 7n - 2^n)$

$= 14^2({}^nC_2 2^{n-2} + {}^nC_3 2^{n-3} \cdot 14 + ... + 14^{n-2})$

$+ (2^n + n2^{n-1} \cdot 2 \cdot 7 - 2^n \cdot 7n - 2^n)$

$= 14^2({}^nC_2 \cdot 2^{n-2} + {}^nC_3 \cdot 2^{n-3} \cdot 14 + ... + 14^{n-2})$

This is divisible by 14^2 i.e. by 196 for all positive integral value of n.

Note : If $n = 1$, ${}^nC_2 = 0$, ${}^nC_3 = 0$ etc.

$\therefore$ Given expression $= 14^2 \times 0 = 0$, which is divisible by 196.

Example – 28

Using binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25 for all positive integers n.

Sol. $6^n - 5n = (1+5)^n - 5n$

$= (1 + {}^nC_1 5 + {}^nC_2 5^2 + ... + {}^nC_n 5^n) - 5n$

$= (1 + n \cdot 5 + {}^nC_2 \cdot 5^2 + ... + {}^nC_n 5^n) - 5n$

$= 1 + {}^nC_2 5^2 + {}^nC_3 5^3 + ... + {}^nC_n 5^n$

$= 1 + 25({}^nC_2 + {}^nC_3 \cdot 5 + ... + {}^nC_n 5^{n-2})$

$= 1 + 25.k$ where k is a positive integer.

$\therefore$ When $6^n - 5n$ is divided by 25, remainder will be 1 for all positive integer n.

Example – 29

Which number is larger, $(1.2)^{4000}$ or 800 ?

Sol. $(1.2)^{4000} = (1+0.2)^{4000}$

$= {}^{4000}C_0 + {}^{4000}C_1(0.2)$ + sum of positive terms

$= 1 + 4000(0.2)$ + a positive number

$= 1 + 800$ + a positive number

> 800

Hence $(1.2)^{4000} > 800$.

Example – 30

The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is

(a) $(n-1)$ (b) $(-1)^n(1-n)$

(c) $(-1)^{n-1}(n-1)^2$ (d) $(-1)^{n-1} n$

Ans. (b)

Sol. $(1+x)(1-x)^n = (1-x)^n + x(1-x)^n$

$\therefore$ Coefficient of x^n is $= (-1)^n + (-1)^{n-1}\,{}^nC_1$

$= (-1)^n[1-n]$

Example – 31

The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is

(a) –132 (b) –144

(c) 132 (d) 144

Ans. (b)

Sol. $(1 - x - x^2 + x^3)^6 = ((1 - x)(1 - x^2))^6 = (1 - x)^6 (1 - x^2)^6$

$= (1 - {}^6C_1 x + {}^6C_2 x^2 - {}^6C_3 x^3 + {}^6C_4 x^4 - {}^6C_5 x^5 + {}^6C_6 x^6)$

$(1 - {}^6C_1 x^2 + {}^6C_2 x^4 - {}^6C_3 x^6 + {}^6C_4 x^8 - {}^6C_5 x^{10} + {}^6C_6 x^{12})$

Coeff. of $x^7 = (-{}^6C_1)(-{}^6C_3) + (-{}^6C_3)({}^6C_2) + (-{}^6C_5)(-{}^6C_1)$

$= 6.20 - 20 . 15 + 6 . 6 = 120 - 300 + 36 = -144$

Example – 32

If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to :

(a) $\left(16, \frac{272}{3}\right)$ (b) $\left(16, \frac{251}{3}\right)$

(c) $\left(14, \frac{251}{3}\right)$ (d) $\left(14, \frac{272}{3}\right)$

Ans. (a)

Sol. On expanding the given expression we get,

$\Rightarrow (1 + ax + bx^2)(1 - {}^{18}C_1 (2x) + {}^{18}C_2 (2x)^2 - {}^{18}C_3 (2x)^3 + ... + {}^{18}C_{18} (2x)^{18})$

Coefficient of x^3,

$\Rightarrow -{}^{18}C_1 (2b) + {}^{18}C_2 (4a) - 8. {}^{18}C_3 = 0$

$\Rightarrow 51a = 3b + 544$ ..(1)

Similarly coefficient of x^4,

$\Rightarrow {}^{18}C_4 (2)^4 - {}^{18}C_3 . 8a + {}^{18}C_2 4b = 0$

$\Rightarrow 32a = 3b + 240$ ..(2)

On solving (1) and (2) we get,

$\Rightarrow$ a = 16 and $b = \frac{272}{3}$

Example – 33

Find the term independent of x in $(1 + x)^m \left(1+\frac{1}{x}\right)^n$

Sol. Given expression $=(1+x)^m\left(1+\frac{1}{x}\right)^n = (1+x)^m\left(\frac{x+1}{x}\right)^n$

$= \frac{(1+x)^{m+n}}{x^n} = x^{-n}(1+x)^{m+n}$

$\therefore$ Required term independent of x

$=$ coefficient of x^0 in $x^{-n}(1 + x)^{m+n}$

$=$ coefficient of x^n in $(1 + x)^{m+n}$

$= {}^{m+n}C_n = \frac{(m+n)!}{n!m!}$

[Since in the expansion of $(1 + x)^n$, co-efficient of $x^r = {}^nC_r$]

Example – 34

Find the coefficient of x^5 in the expansion of the product $(1 + 2x)^6 (1 - x)^7$.

Sol. $(1 + 2x)^6 = [1 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6]$ .....(1)

Again, $(1 - x)^7 = 1 - {}^7C_1 x + {}^7C_2 x^2 - {}^7C_3 x^3 + {}^7C_4 x^4 - {}^7C_5 x^5 + {}^7C_6 x^6 - {}^7C_7 x^7$

$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$ .....(2)

Now $(1 + 2x)^6 (1 - x)^7$

$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + ...)$

$\times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + ...)$

$\therefore$ Required coefficient of x^5 in the product

$= 1 \times (-21) + 12 \times 35 + 60 \times (-35) + 160 \times 21 + 240 \times (-7) + 192 \times 1$

$= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$

Example – 35

Simplify first three terms in the expansion of the following

(i) $(1+2x)^{-4}$ (ii) $(5+4x)^{-1/2}$

Sol. (i) $(1+2x)^{-4}=$

$$1+(-4)(2x)+\frac{(-4)(-4-1)}{2!}(2x)^2+\ldots\ldots\ldots\ldots\ldots$$

$$=1-8x+\frac{(-4)(-5)}{2}(4x^2)+\ldots\ldots\ldots\ldots\ldots$$

$=1-8x+40x^2+\ldots..$

(ii) $$(5+4x)^{\frac{-1}{2}}=5^{\frac{-1}{2}}\left[1+\frac{4x}{5}\right]^{\frac{-1}{2}}$$

$$=5^{\frac{-1}{2}}\left[1+\left(\frac{-1}{2}\right)\left(\frac{4x}{5}\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(\frac{4x}{5}\right)^2+\ldots\right]$$

$$=5^{\frac{-1}{2}}\left[1-\frac{2x}{5}+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}\times\frac{16x^2}{25}+\ldots\right]$$

$$=5^{\frac{-1}{2}}\left[1-\frac{2x}{5}+\frac{6x^2}{25}+\ldots.\right]$$

Example – 36

Find the coefficient of x^4 in the expansion of $\frac{1+x}{1-x}$ if $|x|<1$

Sol. $$\frac{1+x}{1-x}=(1+x)(1-x)^{-1}$$

$$=(1+x)\left[1+\frac{(-1)}{1!}(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3\ldots..\text{to}\infty\right]$$

$=(1+x)(1+x+x^2+x^3+x^4+\ldots\ldots\text{ to }\infty)$

$=[1+x+x^2+x^3+x^4+\ldots\ldots.\text{ to }\infty]+[x+x^2+x^3+x^4+\ldots\ldots\ldots\text{ to }\infty]$

$=1+2x+2x^2+2x^3+2x^4+2x^5+\ldots\ldots\text{ to }\infty$

Hence coefficient of $x^4=2$

Example – 37

Find the square root of 99 correct to 4 places of deicmal.

Sol. $$(99)^{1/2}=(100-1)^{1/2}=\left[100\left(1-\frac{1}{100}\right)\right]^{\frac{1}{2}}$$

$$=\left[100\left(1-\frac{1}{100}\right)\right]^{\frac{1}{2}}$$

$=(100)^{1/2}[1-0.01]^{1/2}$

$$=10\left[1+\frac{\frac{1}{2}}{1!}(-0.01)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.01)^2+\ldots\ldots..to\ \infty\right]$$

$=10[1-0.005-0.0000125+\ldots\ldots\ldots\text{ to }\infty]$

$=10(.9949875)=9.94987=9.9499$

Example – 38

The number of terms in the expansion of $(2x+3y-4z)^n$, is

(a) $n+1$ (b) $n+3$

(c) $\frac{(n+1)(n+2)}{2}$ (d) none of these

Ans. (c)

Sol. Number of terms $= {}^{n+r-1}C_{r-1}$

$= {}^{n+3-1}C_{3-1}$

$$=\frac{(n+2)(n+1)}{2}$$

Example – 39

The coefficient of $(a^3 b^6 c^8 d^9 e f)$ in the expansion of $(a + b + c - d - e - f)^{31}$ is :

(a) 123210 (b) 23110

(c) 3110 (d) none of these

Ans. (d)

Sol. The coefficient of $a^3 b^6 c^8 d^9 e f$ in expansion of

$(a + b + c - d - e - f)^{31}$ is zero as that term is not possible in expansion as sum of powers in not 31.

Example – 40

Find the total number of terms in the expansion of $(1 + a + b)^{10}$ and coefficient of a^2b^3.

Sol. Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $a^2b^3 = \dfrac{10!}{2!\times 3!\times 5!} = 2520$

09

BINOMIAL THEOREM

EXERCISE – 1: Basic Subjective Questions

Section–A (1 Mark Questions)

1. Find the number of terms in the expansion of $(1+x)^{10}+(1-x)^{10}$.

2. What is the value of $\sum_{r=0}^{r=n} 4^r\,{}^nC_r$.

3. Find the general term in the expansion of $\left(2a+\frac{b}{2}\right)^n, n \in N$.

4. Using binomial theorem, write down the expansion of $(1-3x)^7$.

5. Find the 5th term from the end in the expansion of $\left(3x-\frac{1}{x^2}\right)^{10}$.

Section–B (2 Marks Questions)

6. Find the 8th term in the expansion of $\left(x^{\frac{3}{2}}y^{\frac{1}{2}}-x^{\frac{1}{2}}y^{\frac{3}{2}}\right)^{10}$.

7. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then find the value of p.

8. Find the Constant term in the expansion of $\left(x-\frac{1}{x}\right)^{10}$.

9. Does the expansion $\left(2x^2-\frac{1}{x}\right)^{20}$ contain any term involving x^9.

10. Prove that the coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of r^{th} and $(r+1)^{th}$ terms in the expansion of $(1+x)^n$.

11. Find the 4th term from the end in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^8$.

12. Find the term independent of x in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$.

13. If the coefficients of $(2r+1)^{th}$ term and $(r+2)^{th}$ term in the expansion of $(1+x)^{43}$ are equal, find r.

Section–C (3 Marks Questions)

14. Find the middle terms in the expansion of $\left(3x-\frac{2}{x^2}\right)^{15}$.

15. If the coefficient of x in the expansion of $\left(x^2+\frac{p}{x}\right)^5$ is 270. Then find the value of p.

16. Find the middle terms in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{11}$.

17. If the term free from x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, find the value of k.

18. Find the sixth term in the expansion $\left(y^{\frac{1}{2}}+x^{\frac{1}{3}}\right)^n$, if the binomial coefficient of the third term from the end is 45.

19. Find the 4th term from the beginning and 4th term from the end in the expansion of $\left(x+\frac{2}{x}\right)^9$.

20. If n is a positive integer, then prove that $3^{3n}-26n-1$ is divisible by 676.

21. Evaluate $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$.

22. Which term in the expansion of $\left(\left(\frac{x}{\sqrt{y}}\right)^{\frac{1}{3}}+\left(\frac{y}{x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right)^{21}$ contains same powers of x and y.

23. If 'n' is positive integer and three consecutive coefficients in the expansion of $(1+x)^n$ are the ratio 6 : 33 : 110, then 'n' is equal to.

24. Prove that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1\cdot3\cdot5\ldots(2n-1)}{n!}.2^n$.

Section–D (5 Marks Questions)

25. If the seventh term from the beginning and end in the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ are equal, find n.

26. If the coefficients of three consecutive terms in the terms in the expansion of $(1+x)^n$ be 76, 95 and 76, find n.

27. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in A.P., then find the value of n.

28. Using binomial theorem, write down the expansion of $\left(1+2x-3x^2\right)^5$.

29. The coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in A.P., find n.

30. If 3rd, 4th, 5th and 6th terms in the expansion of $(x+\alpha)^n$ be respectively a, b, c and d, then prove that $\frac{b^2-ac}{c^2-bd}=\frac{5a}{3c}$.

EXERCISE – 2: Basic Objective Questions

Section–A (Single Choice Questions)

1. The term independent of x in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$ is.

(a) 2^{nd} (b) 3^{rd}

(c) 4^{th} (d) 5^{th}

2. In the expansion of $\left(\sqrt[3]{\frac{x}{3}}-\sqrt{\frac{3}{x}}\right)^{10}, x>0,$ the constant term is.

(a) –70 (b) 70

(c) 210 (d) –210

3. If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then n is.

(a) 56 (b) 55

(c) 45 (d) 15

4. The coefficient of the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$.

(a) $\frac{5}{4}$ (b) $\frac{7}{4}$

(c) $\frac{9}{4}$ (d) None of these

5. If T_r is the term in the expansion of $(1+x)^{101}$, then the ratio $\frac{T_{20}}{T_{19}}$ equal to.

(a) $\frac{20x}{19}$ (b) $83x$

(c) $19x$ (d) $\frac{83x}{19}$

6. If in the binomial expansion of $(1+x)^n$ where n is a natural number, the coefficients of the 5^{th}, 6^{th} and 7^{th} terms are in A.P., then n is equal to.

(a) 7 or 13 (b) 7 or 14

(c) 7 or 15 (d) 7 or 17

7. The coefficient of the middle term in the expansion of $(2+3x)^4$ is.

(a) 6 (b) 5!

(c) 8! (d) 216

8. If the r^{th} term in the expansion of $\left(\frac{x}{3}-\frac{2}{x^2}\right)^{10}$ contains x^4, then r is equal to.

(a) 2 (b) 3

(c) 4 (d) 5

9. If x^4 occurs in the r^{th} term in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$, then what is the value of r.

(a) 4 (b) 8

(c) 9 (d) 10

10. What is the coefficient of x^3y^4 in $(2x+3y^2)^5$.

(a) 240 (b) 360

(c) 720 (d) 1080

11. If the coefficients of x^7 in $\left[ax^2+\frac{1}{bx}\right]^{11}$ equals the coefficients of x^{-7} in $\left[ax-\left(\frac{1}{bx}\right)\right]^{11}$, then a and b satisfy the relation.

(a) $a-b=1$ (b) $a+b=1$

(c) $\frac{a}{b}=1$ (d) $ab=1$

12. If A and B are coefficients of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ then.

(a) $A=B$ (b) $2A=B$

(c) $A=2B$ (d) $AB=2$

13. What is the coefficient of x^3 in $\frac{(3-2x)}{(1+3x)^3}$.

(a) –272 (b) –540

(c) –870 (d) –918

14. $\sqrt{5}\left[\left(\sqrt{5}+1\right)^{50}-\left(\sqrt{5}-1\right)^{50}\right]$ is.

(a) An irrational number

(b) 0

(c) A natural number

(d) None of these

15. The number of terms in the expansion of $\left[(x+4y)^3(x-4y)^3\right]^2$ is.

(a) 6 (b) 7

(c) 8 (d) 32

16. The term independent of x in the expansion of $\left(\sqrt[6]{x}-\frac{1}{\sqrt[3]{x}}\right)^9$ is.

(a) $-{}^9C_3$ (b) $-{}^9C_4$

(c) $-{}^9C_5$ (d) $-{}^8C_3$

17. If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greater coefficient in the expansion is.

(a) 1594 (b) 792

(c) 924 (d) 2994

18. The coefficient of x^{-7} in the expansion of $\left[ax-\frac{1}{bx^2}\right]^{11}$ will be.

(a) $\frac{462}{b^5}a^6$ (b) $\frac{462a^5}{b^6}$

(c) $\frac{-462a^5}{b^6}$ (d) $\frac{-462a^6}{b^5}$

19. If $(1+x)^n=C_0+C_1x+C_2x^2+.....+C_nx^n$, then, the value of

$$\frac{(C_0+C_1)(C_1+C_2).........(C_{n-1}+C_n)}{C_0C_1C_2.......C_{n-1}}$$ is.

(a) $\frac{(n+3)^3}{(2n)!}$ (b) $\frac{(n+1)^n}{n!}$

(c) $\frac{(2n)!}{(n+1)!}$ (d) $\frac{(n-1)^n}{n!}$

20. If the coefficients of r^{th} and $(r+1)^{th}$ terms in the expansion of $(3+7x)^{29}$ are equal, then the value of r is.

(a) 31 (b) 11

(c) 18 (d) 21

Section–B (Case Study Questions)

Case Study–1

21. In the binomial expansion of $(a+b)^n$, the general term of an expansion is $T_{r+1}={}^nC_ra^{n-r}b^r$, the middle term of an expansion is $\left(\frac{n}{2}+1\right)^{th}$ term when 'n' is even and the middle terms of an expansion are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term when 'n' is odd.

Based on the above information. Answer the following questions.

(i) If the co-efficient of x^7 and x^8 in the expansion of are equal, then the value of n is.

(a) 56 (b) 55

(c) 47 (d) 19

(ii) The term independent of 'x' in the expansion of $\left(x^3-\frac{1}{x^2}\right)^{15}$ is.

(a) ${}^{15}C_9$ (b) 0

(c) $-\left({}^{15}C_9\right)$ (d) ${}^{15}C_{11}$

(iii) If 'a' is a real number and if the middle term of $\left(\frac{a}{3}+3\right)^8$ is 1120, then value of 'a' is.

(a) ± 2 (b) ± 1

(c) $\pm\sqrt{3}$ (d) $\pm\sqrt{2}$

(iv) The co-efficient of the middle term in the binomial expansion of $(1+\alpha x)^4$ and $(1-\alpha x)^6$ is the same, if α equals.

(a) $\frac{-3}{10}$ (b) $\frac{10}{3}$

(c) $\frac{-5}{3}$ (d) $\frac{3}{5}$

Case Study–2

22. For the binomial expansion $\left(3x^2-\frac{1}{2x^3}\right)^{10}$, Answer the following questions.

(i) The term independent of x when multiplied by 8 is.

(a) 76455 (b) 76465

(c) 76545 (d) 76655

(ii) Middle term is equal to.

(a) $\frac{15309}{8}x^5$ (b) $\frac{-15309}{8}x^5$

(c) $-\frac{15309}{8x^5}$ (d) $\frac{15309}{8x^5}$

(iii) The coefficient of the term with x^{10} is.

(a) $\frac{5\times 3^{10}}{4}$ (b) $\frac{5\times 3^{8}}{4}$

(c) $\frac{5\times 3^{10}}{8}$ (d) $\frac{5\times 3^{8}}{8}$

(iv) The 5^{th} term from the end is equal to.

(a) $\frac{17010}{32x^{10}}$ (b) $\frac{8505}{32}x^{10}$

(c) $\frac{17010}{32}x^{10}$ (d) $\frac{8505}{32x^{10}}$

Case Study–3

23. $(1-2x)^7$ and $\left(2x^2-\frac{1}{x}\right)^8$ are two binomial expansions having the index 7 and 8 respectively. Based on the above information answer the following questions.

(i) 4^{th} term of the expansion $\left(2x^2-\frac{1}{x}\right)^8$ is.

(a) $-1892x^7$ (b) $-1792x^5$

(c) $192x^8$ (d) $-1792x^7$

(ii) The sum of the co-efficient of middle terms of the expansion of $(1-2x)^7$ is.

(a) 280 (b) 180

(c) 560 (d) –382

(iii) The term independent of x in the expansion of $\left(2x^2-\frac{1}{x}\right)^8$ is.

(a) 2 (b) 1

(c) –2 (d) 0

(iv) The 6^{th} term from the end of the expansion of $(1-2x)^7$.

(a) ${}^7C_5.16x^2$ (b) ${}^7C_3x^3$

(c) $-{}^7C_5.4x^2$ (d) ${}^7C_5 4x^2$

Section–C (Assertion & Reason Type Questions)

24. **Assertion:** Let L be the set of all lines in a plane and R be the relation in L defined as $R=\{(L_1,L_2): L_1 \text{ is perpendicular to } L_2\}$. This relation is not equivalence relation.

Reason: A relation is said to be equivalence relation if it is reflexive, symmetric and transitive.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

25. **Assertion:** $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)....\left(1+\frac{C_n}{C_{n-1}}\right)=\frac{(n+1)^n}{n!}$.

Reason: ${}^nC_r=\frac{n(n-1)....(n-r+1)}{r(r-1)....1}$.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

26. **Assertion:** The r^{th} term from the end in the expansion of $(x+a)^n$ is ${}^nC_{n-r+1}x^{r-1}a^{n-r+1}$.

Reason: The r^{th} term from the end in the expansion of $(x+a)^n$ is $(n-r+2)^{th}$ term.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

27. **Assertion:** In the expansion of $(x+2y)^8$, the middle term is 4^{th} term.

Reason: If n is even in the expansion of $(a+b)^n$, then $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

28. **Assertion:** General term of the expansion $(x+2y)^9$ is ${}^9C_r 2^r.x^{9-r}.y^r$.

Reason: General term of the expansion $(x+a)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r}.a^r$.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

29. **Assertion:** Number of terms in the expansion of $\left(\sqrt{x}+\sqrt{y}\right)^{10}+\left(\sqrt{x}-\sqrt{y}\right)^{10}$ is 6.

Reason: If n is even, then the expansion of $\left\{(x+a)^n+(x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

30. **Assertion:** The coefficients of the expansions are arranged in an array. This array is called Pascal's triangle.

Reason: There are 11^{th} terms in the expansion of $(4x+7y)^{10}+(4x-7y)^{10}$.

(a) Both Assertion and Reason are correct; Reason is the correct explanation of Assertion.

(b) Both Assertion and Reason are correct; Reason is not the correct explanation of Assertion.

(c) Assertion is correct; Reason is incorrect.

(d) Reason is correct, but Assertion is incorrect.

Answer Key

CHAPTER 9 : BINOMIAL THEOREM

EXERCISE-1:

1. 6 **2.** 5^n **3.** $T_{r+1} = 2^{n-2r}\ {}^nC_r a^{n-r} b^r$

4. $1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7$

5. $\frac{17010}{x^8}$ **6.** $-120x^8y^{12}$ **7.** $p = \pm 2$ **8.** -252

9. There is no term with x^9 in the given expansion.

11. $T_4 = \frac{-2800}{x^2}$ **12.** $T_4 = \frac{1792}{9}$

13. $r = 14$ **14.** $T_8 = -715.2^7.3^{10}x^{-6}$, $T_9 = 715.2^8.3^9x^{-9}$

15. $p = 3$ **16.** $T_6 = -462x^9$, $T_7 = 462x^2$

17. $k = \pm 3$ **18.** $T_6 = 252y^{\frac{5}{2}}x^{\frac{5}{3}}$ **19.** $672x^3$ & $\frac{5376}{x^3}$

21. $64x^6 - 96x^4 + 36x^2 - 2$ **22.** 10^{th} term **23.** $n = 12$

25. $n = 12$ **26.** $n = 8$ **27.** $n = \frac{7}{2}$ or 1

28. $1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$

29. $n = 7$ or 14.

EXERCISE-2:

1. (c) **2.** (c) **3.** (b) **4.** (a) **5.** (d)

6. (b) **7.** (d) **8.** (b) **9.** (c) **10.** (c)

11. (d) **12.** (c) **13.** (d) **14.** (c) **15.** (b)

16. (a) **17.** (c) **18.** (b) **19.** (b) **20.** (b)

21. **(i)** (b) **(ii)** (c) **(iii)** (a) **(iv)** (a)

22. **(i)** (c) **(ii)** (c) **(iii)** (a) **(iv)** (d)

23. **(i)** (d) **(ii)** (a) **(iii)** (d) **(iv)** (c)

24. (a) **25.** (b) **26.** (a) **27.** (d) **28.** (a)

29. (a) **30.** (c)

EXERCISE – 1: Basic Subjective Questions

Section–A (1 Mark Questions)

1. The expansion of $(1+x)^{10}+(1-x)^{10}$ is given as

$$\left[{}^{10}C_0x^0+{}^{10}C_1x^1+{}^{10}C_2x^2+...+{}^{10}C_{10}x^{10}\right]+$$
$$\left[{}^{10}C_0(-x)^0+{}^{10}C_1(-x)^1+{}^{10}C_2(-x)^2+...+{}^{10}C_{10}(-x)^{10}\right]$$
$$=2\left[{}^{10}C_0x^0+{}^{10}C_2x^2+{}^{10}C_4x^4+...+{}^{10}C_{10}x^{10}\right]$$

Hence, the number of terms in the given expansion is 6.

2.

$$\sum_{r=0}^{r=n}4^r\,{}^nC_r=4^0.{}^nC_0+4^1.{}^nC_1+4^2.{}^nC_2+...+4^n.{}^nC_n$$
$$={}^nC_0.4^0+{}^nC_1.4^1+{}^nC_2.4^2+...+{}^nC_n.4^n$$
$$=(1+4)^n=5^n.$$

3. The general term in the expansion of $\left(2a+\frac{b}{2}\right)^n$ is –

$$T_{r+1}={}^nC_r.(2a)^{n-r}\left(\frac{b}{2}\right)^r=2^{n-2r}.{}^nC_r.a^{n-r}.b^r$$

4. The binomial expansion of $(1-3x)^7$ is given by

$${}^7C_0(3x)^0-{}^7C_1(3x)^1+{}^7C_2(3x)^2-{}^7C_3(3x)^3+{}^7C_4(3x)^4$$
$$-{}^7C_5(3x)^5+{}^7C_6(3x)^6-{}^7C_7(3x)^7$$
$$=1-21x+189x^2-945x^3+2835x^4$$
$$-5103x^5+5103x^6-2187x^7.$$

5. The 5^{th} term from the end in $\left(3x-\frac{1}{x^2}\right)^{10}$ $=5^{th}$ term from the beginning in $\left(\frac{1}{x^2}-3x\right)^{10}$ which is given by

$$T_5=T_{4+1}={}^{10}C_4\left(\frac{1}{x^2}\right)^{10-4}(-3x)^4$$
$$=3^4.{}^{10}C_4.x^{-8}$$
$$=\frac{10\times9\times8\times7\times81}{4\times3\times2\times1\times x^8}$$
$$=\frac{17010}{x^8}.$$

Section–B (2 Marks Questions)

6. The 8^{th} term in the expansion of $\left(x^{\frac{3}{2}}y^{\frac{1}{2}}-x^{\frac{1}{2}}y^{\frac{3}{2}}\right)^{10}$ is given by-

$$\because\quad T_8=T_{7+1}$$
$$T_8=T_{7+1}={}^{10}C_7\left(x^{\frac{3}{2}}y^{\frac{1}{2}}\right)^{10-7}\left(-x^{\frac{1}{2}}y^{\frac{3}{2}}\right)^7$$
$$=-\frac{10!}{7!\times3!}.x^{\frac{9}{2}+\frac{7}{2}}.y^{\frac{3}{2}+\frac{21}{2}}$$
$$=-\frac{10\times9\times8}{3\times2}x^8y^{12}$$
$$=-120x^8y^{12}.$$

7. In the binomial expansion of $\left(\frac{p}{2}+2\right)^8$, we observe that $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term is the middle term.
It is given that the middle term is 1120.
$\therefore\ T_5 = 1120$

$\Rightarrow {}^8C_4\left(\frac{p}{2}\right)^{8-4}(2)^4 = 1120$

$\Rightarrow \frac{8!}{4!\times 4!}p^4 = 1120$

$\Rightarrow p^4 = 16$
$\Rightarrow p = \pm 2$
Hence, the real values of p is ± 2.

8. The general term for the expansion of $\left(x-\frac{1}{x}\right)^{10}$ can be given by

$T_{r+1} = {}^{10}C_r(x)^{10-r}\left(-\frac{1}{x}\right)^r$

$= (-1)^r\ {}^{10}C_r(x)^{10-2r}$

For constant term,
$10-2r = 0$
$\Rightarrow r = 5$
Therefore,
$T_{5+1} = {}^{10}C_5(-1)^5 = -252$
Hence, the constant term is -252.

9. Suppose x^9 occurs in the given expression $\left(2x^2-\frac{1}{x}\right)^{20}$ at the $(r+1)^{th}$ term.
Then, we have

$T_{r+1} = {}^{20}C_r\left(2x^2\right)^{20-r}\left(-\frac{1}{x}\right)^r$

$= (-1)^r.{}^{20}C_r(2)^{20-r}(x)^{40-2r-r}$

For this term to contain x^9, we must have
$40-3r = 9$
$\Rightarrow 3r = 31$
$\Rightarrow r = \frac{31}{3}$
It is not possible, as r is not an integer.
Hence, there is no term with x^9 in the given expression.

10. Coefficient of the $(r+1)^{th}$ term in $(1+x)^{n+1}$ is ${}^{n+1}C_r$.

Coefficient of the r^{th} term in $(1+x)^n$ is ${}^nC_{r-1}$.

Coefficient of the $(r+1)^{th}$ term in $(1+x)^n$ is ${}^nC_r$.

Therefore,
Sum of the coefficients of the r^{th} and $(r+1)^{th}$ terms in $(1+x)^n = {}^nC_{r-1} + {}^nC_r$

$= {}^{n+1}C_r \qquad \left[\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r\right]$

Hence proved.

11. Let T_4 be the 4^{th} term from the end of the given expression,
Then,

4^{th} term from the end in $\left(\frac{4x}{5}-\frac{5}{2x}\right)^8 = 4^{th}$ term from the beginning in $\left(\frac{5}{2x}-\frac{4x}{5}\right)^8$

Thus, we have

$T_4 = T_{3+1} = {}^8C_3\left(\frac{5}{2x}\right)^{8-3}\left(-\frac{4x}{5}\right)^3$

$$= -{}^8C_3\left(\frac{5}{2}\right)^5\left(\frac{4}{5}\right)^3 x^{-2}$$

$$= -{}^8C_3 . 5^2 . 2^1 x^{-2}$$

$$= -\frac{8!}{3!\times 5!}.\frac{50}{x^2}$$

$$= -\frac{8.7.6}{3.2}.\frac{50}{x^2}$$

$$= -\frac{2800}{x^2}.$$

12. Suppose the $(r+1)^{th}$ term in the given expansion $\left(2x+\frac{1}{3x^2}\right)^9$ is independent of *x*.

Thus,

$$T_{r+1} = {}^9C_r\,(2x)^{9-r}\left(\frac{1}{3x^2}\right)^r = {}^9C_r.\frac{2^{9-r}}{3^r}x^{9-3r}$$

For this term to be independent of *x*, we must have

$9-3r=0$

$\Rightarrow r=3$

Hence, the required term is the 4^{th} term which can be written as

$$T_4 = T_{3+1} = {}^9C_3\frac{2^{9-3}}{3^3} = {}^9C_3\times\frac{2^6}{3^3}$$

$$= \frac{9!}{3!6!}\times\frac{2^6}{3^3} = \frac{1792}{9}.$$

13. We know that the coefficient of the r^{th} term in the expansion of $(1+x)^n$ is ${}^nC_{r-1}$.

Therefore, the coefficients of the $(2r+1)^{th}$ and $(r+2)^{th}$ terms in the given expression are ${}^{43}C_{2r}$ and ${}^{43}C_{r+1}$.

For these coefficients to be equal,

we must have,

$2r = r+1$ or $2r+r+1 = 43$

$\left[\because\ {}^nC_r = {}^nC_s \Rightarrow r=s \text{ or } r+s=n\right]$

$\Rightarrow r=1$ or $r=14$

$\Rightarrow r=14$ [$\because$ for $r=1$ it gives the same term].

Section–C (3 Marks Questions)

14. Given $n=15$, which is an odd number.

Thus, the middle terms are

$\left(\frac{15+1}{2}\right)^{th}$ and $\left(\frac{15+1}{2}+1\right)^{th}$ i.e. 8^{th} and 9^{th} terms.

Now, $T_8 = T_{7+1}$

$$= {}^{15}C_7\,(3x)^{15-7}\left(\frac{-2}{x^2}\right)^7$$

$$= -\frac{15!}{7!\times 8!}(3x)^8\left(\frac{2}{x^2}\right)^7$$

$$= -\frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2}\times3^8\times2^7x^{8-14}$$

$$= -13\times11\times5\times9\times3^8\times2^7x^{8-14}$$

$$= -715.2^7.3^{10}.x^{-6}$$

And, $T_9 = T_{8+1}$

$$= {}^{15}C_8\,(3x)^{15-8}\left(\frac{-2}{x^2}\right)^8$$

$$= \frac{15!}{8!\times 7!}(3x)^7\left(\frac{2}{x^2}\right)^8$$

$$= \frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2}\times3^7\times2^8\times x^{7-16}$$

$$= 715.2^8.3^9.x^{-9}.$$

15. The general term for the expansion of $\left(x^2+\frac{p}{x}\right)^5$ can be given by

$T_{r+1} = {}^5C_r\left(x^2\right)^{5-r}\left(\frac{p}{x}\right)^r = {}^5C_r\left(x\right)^{10-3r}\left(p\right)^r$

For coefficient of x,

$10-3r=1$

$\Rightarrow r=3$

Therefore, the coefficient of $x = {}^5C_3\left(p\right)^3$

$= \frac{5!}{3!\times 2!}p^3 = 10p^3$

Since, coefficient of x is 270,

$\Rightarrow 10p^3 = 270$

$\Rightarrow p^3 = 27$

$\Rightarrow p=3$.

16. Given expansion is $\left(x^4-\frac{1}{x^3}\right)^{11}$,

Here, $n=11$, which is an odd number.

Thus, the middle terms are

$\left(\frac{11+1}{2}\right)^{th}$ and $\left(\frac{11+1}{2}+1\right)^{th}$ i.e. 6^{th} and 7^{th}.

Now,

$T_6 = T_{5+1}$

$= {}^{11}C_5\left(x^4\right)^{11-5}\left(-\frac{1}{x^3}\right)^5$

$= -\frac{11\times10\times9\times8\times7}{5\times4\times3\times2}\times\left(x\right)^{24-15}$

$= -462x^9$

And,

$T_7 = T_{6+1}$

$= {}^{11}C_6\left(x^4\right)^{11-6}\left(-\frac{1}{x^3}\right)^6$

$= \frac{11\times10\times9\times8\times7}{5\times4\times3\times2}\left(x\right)^{20-18}$

$= 462x^2$.

17. Let $\left(r+1\right)^{th}$ term in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ be free from x and be equal to 405. Then,

$T_{r+1} = {}^{10}C_r\left(\sqrt{x}\right)^{10-r}\left(-\frac{k}{x^2}\right)^r$

$= {}^{10}C_r x^{5-\frac{5r}{2}}\left(-k\right)^r \ldots(1)$

If T_{r+1} is independent of x, then

$5-\frac{5r}{2}=0 \Rightarrow r=2$

Putting $r=2$ in (1).

We obtain $T_3 = {}^{10}C_2\left(-k\right)^2 = 45k^2$

But it is given that the term free from x is equal to 405.

$\therefore 45k^2 = 405$

$\Rightarrow k^2 = 9$

$\Rightarrow k = \pm 3$

Hence, the value of k is ± 3.

18. The third term from the end in the expansion of $\left(y^{\frac{1}{2}}+x^{\frac{1}{3}}\right)^n$, is the third term from the beginning in the expansion of $\left(x^{\frac{1}{3}}+y^{\frac{1}{2}}\right)^n$.

$\therefore$ The binomial coefficient of the third term from the end $= {}^nC_2$

It is given that the binomial coefficient of the third term from the end is 45.

$\therefore {}^nC_2 = 45$

$\Rightarrow \frac{n\left(n-1\right)}{2} = 45$

$\Rightarrow n^2-n-90=0$

$\Rightarrow (n-10)(n+9)=0$

$\Rightarrow n=10$ ($\because$ n cannot be negative)

Let T_6 be the sixth term in the binomial expansion of $\left(y^{\frac{1}{2}}+x^{\frac{1}{3}}\right)^n$. Then

$$T_6={}^{10}C_5\left(y^{\frac{1}{2}}\right)^{10-5}\left(x^{\frac{1}{3}}\right)^5={}^{10}C_5 y^{\frac{5}{2}}x^{\frac{5}{3}}=252y^{\frac{5}{2}}x^{\frac{5}{3}}$$

19. 4^{th} term from the beginning in the expansion of $\left(x+\frac{2}{x}\right)^9=T_4=T_{3+1}$

$$\therefore T_4={}^9C_3\left(x^{9-3}\right)\left(\frac{2}{x}\right)^3=\frac{9\times8\times7}{3\times2}\left(x^6\right)\left(\frac{8}{x^3}\right)=672x^3$$

Let T_4 be the 4^{th} term from the end.

Since, 4^{th} term from the end in $\left(x+\frac{2}{x}\right)^9$ =

4^{th} term from the beginning in $\left(\frac{2}{x}+x\right)^9$.

Thus,

$$T_4=T_{3+1}={}^9C_3\left(\frac{2}{x}\right)^{9-3}(x)^3$$

$$=\frac{9\times8\times7}{3\times2}\left(\frac{64}{x^6}\right)(x)^3=\frac{5376}{x^3}$$

20. Given,

$$3^{3n}-26n-1=27^n-26n-1$$

$$=(1+26)^n-26n-1$$

$$={}^nC_0\times26^0+{}^nC_1\times26^1+{}^nC_2\times26^2+{}^nC_3\times26^3+....+{}^nC_n\times26^n-26n-1$$

$$=1+26n+{}^nC_2\times26^2+{}^nC_3\times26^3+....+{}^nC_n\times26^n-26n-1$$

$$={}^nC_2\times26^2+{}^nC_3\times26^3+....+{}^nC_n\times26^n$$

$$=26^2\left[{}^nC_2+{}^nC_3\times26^1+{}^nC_4\times26^2+....+{}^nC_n\times26^{n-2}\right]$$

$$=676\left[{}^nC_2+{}^nC_3\times26^1+{}^nC_4\times26^2+....+{}^nC_n\times26^{n-2}\right]$$

Hence, $3^{3n}-26n-1$ is divisible by 676.

21. Binomial expansion of $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$ is given by

$$2\left[{}^6C_0x^6\left(\sqrt{x^2-1}\right)^0+{}^6C_2x^4\left(\sqrt{x^2-1}\right)^2+{}^6C_4x^2\left(\sqrt{x^2-1}\right)^4+{}^6C_6x^0\left(\sqrt{x^2-1}\right)^6\right]$$

$$=2\left[x^6+15x^4\left(x^2-1\right)+15x^2\left(x^2-1\right)^2+\left(x^2-1\right)^3\right]$$

$$=2\left[x^6+15x^6-15x^4+15x^2\left(x^4-2x^2+1\right)+\left(x^6-1+3x^2-3x^4\right)\right]$$

$$=2\left[x^6+15x^6-15x^4+15x^6-30x^4+15x^2+x^6-1+3x^2-3x^4\right]$$

$$=64x^6-96x^4+36x^2-2.$$

22. Suppose T_{r+1} term in the given expansion of $\left(\left(\frac{x}{\sqrt{y}}\right)^{\frac{1}{3}}+\left(\frac{y}{x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right)^{21}$ contains x and y to one and the same power.

Thus,

$$T_{r+1}={}^{21}C_r\left[\left(\frac{x}{\sqrt{y}}\right)^{\frac{1}{3}}\right]^{21-r}\left[\left(\frac{y}{x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right]^r$$

$$={}^{21}C_r\left(\frac{x^{\frac{(21-r)}{3}}}{x^{\frac{r}{6}}}\right)\left(\frac{y^{\frac{r}{2}}}{y^{\left(\frac{21-r}{6}\right)}}\right)$$

$$={}^{21}C_r\,(x)^{7-\frac{r}{2}}\,(y)^{\frac{2r}{3}-\frac{7}{2}}$$

Now, if x and y have the same power, then

$$7-\frac{r}{2}=\frac{2r}{3}-\frac{7}{2}$$

$$\Rightarrow\frac{2r}{3}+\frac{r}{2}=7+\frac{7}{2}$$

$$\Rightarrow\frac{7r}{6}=\frac{21}{2}$$

$$\Rightarrow r=9$$

Hence, the required term is the 10^{th} term.

23. Let the consecutive coefficient of $(1+x)^n$ are ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$

From the given condition,

${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 6:33:110$

Now, ${}^nC_{r-1} : {}^nC_r = 6:33$

$$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{6}{33}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{2}{11}$$

$11r = 2n - 2r + 2$

$\Rightarrow 2n - 13r + 2 = 0 \quad .....(i)$

and ${}^nC_r : {}^nC_{r+1} = 33:110$

$$\Rightarrow \frac{n!}{r!(n-r)!} \times \frac{(r+1)!(n-r-1)!}{n!} = \frac{33}{110} = \frac{3}{10}$$

$$\Rightarrow \frac{(r+1)}{n-r} = \frac{3}{10}$$

$\Rightarrow 3n - 13r - 10 = 0 \quad .....(ii)$

Solving (i) and (ii),

we get, $n = 12$.

24. Suppose the term independent of x is the $(r+1)^{th}$ term.

$$\therefore T_{r+1} = {}^{2n}C_r x^{2n-r} \left(\frac{1}{x}\right)^r = {}^{2n}C_r x^{2n-2r}$$

For this term to be independent of x,

we must have:

$2n - 2r = 0$

$\Rightarrow n = r$

$$\therefore \text{ Required coefficient } = {}^{2n}C_n = \frac{(2n)!}{(n!)^2}$$

$$= \frac{\{1\cdot3\cdot5...(2n-3)(2n-1)\}\{2\cdot4\cdot6...(2n-2)(2n)\}}{(n!)^2}$$

$$= \frac{\{1\cdot3\cdot5...(2n-3)(2n-1)\}2^n}{n!}.$$

Section–D (5 Marks Questions)

25. The general term in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ is given by

$$T_{r+1} = {}^nC_r \left(\sqrt[3]{2}\right)^{n-r} \left(\frac{1}{\sqrt[3]{3}}\right)^6 = {}^nC_r \times 2^{\frac{n-r}{3}} \times \left(\frac{1}{3}\right)^{\frac{r}{3}}$$

Therefore, 7^{th} term from the beginning is given by

$$T_7 = T_{6+1} = {}^nC_6 \times 2^{\frac{n}{3}-2} \times \left(\frac{1}{3}\right)^2$$

Since 7^{th} term from the end of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$

$= 7^{th}$ term (t_7) from the beginning of $\left(\frac{1}{\sqrt[3]{3}} + \sqrt[3]{2}\right)^n$.

Therefore,

$$t_7 = t_{6+1} = {}^nC_6 \left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} \left(\sqrt[3]{2}\right)^6$$

$$= {}^nC_6 \times \left(\frac{1}{3}\right)^{\frac{n}{3}-2} \times 2^2$$

It is given that, $T_7 = t_7$

$$\Rightarrow {}^nC_6 \times 2^{\frac{n}{3}-2} \times \left(\frac{1}{3}\right)^2 = {}^nC_6 \times \left(\frac{1}{3}\right)^{\frac{n}{3}-2} \times 2^2$$

$$\Rightarrow \frac{2^{\frac{n}{3}-2}}{2^2} = \frac{3^2}{3^{\frac{n}{3}-2}}$$

$$\Rightarrow (6)^{\frac{n}{3}-2} = 6^2$$

$$\Rightarrow \frac{n}{3} - 2 = 2$$

$\Rightarrow n = 12$

Hence, the value of n is 12.

26. Suppose $r,(r+1)$ and $(r+2)$ are three consecutive terms in the given expansion.
The coefficients of these terms will be
${}^nC_{r-1}, {}^nC_r$ and ${}^nC_{r+1}$
According to the question,

$${}^nC_{r-1}=76$$

$${}^nC_r=95$$

$${}^nC_{r+1}=76$$

$$\Rightarrow {}^nC_{r-1}={}^nC_{r+1}$$

$$\Rightarrow r-1+r+1=n\left[\because {}^nC_r={}^nC_s \Rightarrow r=s \text{ or } r+s=n\right]$$

$$\Rightarrow r=\frac{n}{2}$$

Also, $\dfrac{{}^nC_r}{{}^nC_{r-1}}=\dfrac{95}{76}$

$$\Rightarrow \frac{n-r+1}{r}=\frac{95}{76}$$

$$\Rightarrow \frac{\frac{n}{2}+1}{\frac{n}{2}}=\frac{95}{76}\left[\because r=\frac{n}{2}\right]$$

$$\Rightarrow 38n+76=\frac{95n}{2}$$

$$\Rightarrow \frac{\frac{n}{2}+1}{\frac{n}{2}}=\frac{95}{76}\left[\because r=\frac{n}{2}\right]$$

$$\Rightarrow 38n+76=\frac{95n}{2}$$

$$\Rightarrow \frac{19n}{2}=76$$

$$\Rightarrow n=8.$$

27. Given: $(1+x)^{2n}$

Thus, we have;

$$T_2=T_{1+1}={}^{2n}C_1x^1$$

$$T_3=T_{2+1}={}^{2n}C_2x^2$$

$$T_4=T_{3+1}={}^{2n}C_3x^3$$

We have coefficients of the 2^{nd}, 3^{rd} and 4^{th} terms in AP.

$$\therefore 2\left({}^{2n}C_2\right)={}^{2n}C_1+{}^{2n}C_3$$

$$\Rightarrow 2=\frac{{}^{2n}C_1}{{}^{2n}C_2}+\frac{{}^{2n}C_3}{{}^{2n}C_2}$$

$$\Rightarrow 2=\frac{2}{2n-1}+\frac{2n-2}{3}$$

$$\Rightarrow 12n-6=6+4n^2-4n-2n+2$$

$$\Rightarrow 4n^2-18n+14=0$$

$$\Rightarrow 2n^2-9n+7=0$$

$$\Rightarrow 2n^2-7n-2n+7=0$$

$$\Rightarrow n(2n-7)-1(2n-7)=0$$

$$\Rightarrow 2n-7=0 \text{ or } n-1=0$$

$$\Rightarrow n=\frac{7}{2} \text{ or } n=1.$$

28. Consider $(1+2x)$ and $3x^2$ as two separate entities and apply the binomial theorem.

Now, $\left(1+2x-3x^2\right)^5$

$$={}^5C_0(1+2x)^5-{}^5C_1(1+2x)^4\left(3x^2\right)^1+{}^5C_2(1+2x)^3\left(3x^2\right)^2$$

$$-{}^5C_3(1+2x)^2\left(3x^2\right)^3+{}^5C_4(1+2x)^1\left(3x^2\right)^4-{}^5C_5\left(3x^2\right)^5$$

$$=(1+2x)^5-5(1+2x)^4\times 3x^2+10\times(1+2x)^3\times 9x^4$$

$$-10\times(1+2x)^2\times 27x^6+5(1+2x)\times 81x^8-243x^{10}$$

$$=\left[{}^5C_0\times(2x)^0+{}^5C_1\times(2x)^1+{}^5C_2\times(2x)^2+{}^5C_3\times(2x)^3+{}^5C_4\times(2x)^4+{}^5C_5\times(2x)^5\right]$$

$$-15x^2\left[{}^4C_0(2x)^0+{}^4C_1(2x)^1+{}^4C_2(2x)^2+{}^4C_3(2x)^3+{}^4C_4(2x)^4\right]$$

$$+90x^4\left[1+8x^3+6x+12x^2\right]-270x^6\left(1+4x^2+4x\right)+405x^8+810x^9-243x^{10}$$

$$=\left[1+10x+40x^2+80x^3+80x^4+32x^5\right]$$

$$-\left[15x^2+120x^3+360x^4+480x^5+240x^6\right]$$

$$+90x^4+720x^7+540x^5+1080x^6-270x^6-1080x^8$$

$$-1080x^7+405x^8+810x^9-243x^{10}$$

$= 1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6$
$- 360x^7 - 675x^8 + 810x^9 - 243x^{10}$.

29. Coefficients of the 5^{th} , 6^{th} and 7^{th} terms in the given expansion $(1+x)^n$ are ${}^nC_4, {}^nC_5$ and ${}^nC_6$.

These coefficients are in AP.

Thus, we have

$$2\,{}^nC_5 = {}^nC_4 + {}^nC_6$$

On dividing both sides by ${}^nC_5$, we get

$$2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5}$$

$$\Rightarrow 2 = \frac{\frac{n!}{4!(n-4)!}}{\frac{n!}{5!(n-5)!}} + \frac{\frac{n!}{6!(n-6)!}}{\frac{n!}{5!(n-5)!}}$$

$$\Rightarrow 2 = \frac{5!(n-5)!}{4!(n-4)!} + \frac{5!(n-5)!}{6!(n-6)!}$$

$$\Rightarrow 2 = \frac{5}{n-4} + \frac{n-5}{6}$$

$$\Rightarrow 12(n-4) = 30 + (n-5)(n-4)$$

$$\Rightarrow 12n - 48 = 30 + n^2 - 4n - 5n + 20$$

$$\Rightarrow n^2 - 21n + 98 = 0$$

$$\Rightarrow (n-14)(n-7) = 0$$

$$\Rightarrow n = 7 \text{ or } 14.$$

30. Sol: The 3^{rd}, 4^{th}, 5^{th} and 6^{th} terms in the expansion of $(x+\alpha)^n$ are ${}^nC_2x^{n-2}\alpha^2$, ${}^nC_3x^{n-3}\alpha^3$, ${}^nC_4x^{n-4}\alpha^4$ and ${}^nC_5x^{n-5}\alpha^5$, respectively.

Now, ${}^nC_2x^{n-2}\alpha^2 = a$

$${}^nC_3x^{n-3}\alpha^3 = b$$

$${}^nC_4x^{n-4}\alpha^4 = c$$

$${}^nC_5x^{n-5}\alpha^5 = d$$

Therefore,

$$\frac{a}{b} = \frac{{}^nC_2x^{n-2}\alpha^2}{{}^nC_3x^{n-3}\alpha^3} = \left(\frac{3}{n-2}\right)\left(\frac{x}{\alpha}\right)$$

$$\frac{b}{c} = \frac{{}^nC_3x^{n-3}\alpha^3}{{}^nC_4x^{n-4}\alpha^4} = \left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)$$

$$\frac{c}{d} = \frac{{}^nC_4x^{n-4}\alpha^4}{{}^nC_5x^{n-5}\alpha^5} = \left(\frac{5}{n-4}\right)\left(\frac{x}{\alpha}\right)$$

$$LHS = \frac{b^2 - ac}{c^2 - bd}$$

$$= \left(\frac{b}{c}\right)^2 \left(\frac{1 - \left(\frac{a}{b}\right)\left(\frac{c}{b}\right)}{1 - \left(\frac{b}{c}\right)\left(\frac{d}{c}\right)}\right)$$

$$= \left[\left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)\right]^2 \left(\frac{1 - \left(\frac{3}{n-2}\right)\left(\frac{x}{\alpha}\right)\left(\frac{n-3}{4}\right)\left(\frac{\alpha}{x}\right)}{1 - \left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)\left(\frac{n-4}{5}\right)\left(\frac{\alpha}{x}\right)}\right)$$

$$= \left[\left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)\right]^2 \left(\frac{1 - \left(\frac{3}{n-2}\right)\left(\frac{n-3}{4}\right)}{1 - \left(\frac{4}{n-3}\right)\left(\frac{n-4}{5}\right)}\right)$$

$$= \left[\left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)\right]^2 \left(\frac{\frac{4n-8-3n+9}{4(n-2)}}{\frac{5n-15-4n+16}{5(n-3)}}\right)$$

$$= \left[\left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)\right]^2 \left(\frac{5(n-3)}{4(n-2)}\right)$$

$$= \left[\left(\frac{4}{n-3}\right)\left(\frac{x}{\alpha}\right)\right]^2 \left(\frac{5(n-3)}{4(n-2)}\right)$$

$$= \left(\frac{20}{(n-2)(n-3)}\right)\left(\frac{x}{\alpha}\right)^2$$

$$\text{RHS} = \frac{5a}{3c}$$

$$= \frac{5}{3}\frac{{}^{n}C_2 x^{n-2}\alpha^2}{{}^{n}C_4 x^{n-4}\alpha^4}$$

$$= \frac{5}{3}\frac{{}^{n}C_2}{{}^{n}C_4}\left(\frac{x}{\alpha}\right)^2$$

$$= \frac{5}{3}.\frac{\frac{n!}{2!(n-2)!}}{\frac{n!}{4!(n-4)!}}\left(\frac{x}{\alpha}\right)^2$$

$$= \frac{5}{3}.\frac{4!(n-4)!}{2!(n-2)!}\left(\frac{x}{\alpha}\right)^2$$

$$= \left(\frac{20}{(n-2)(n-3)}\right)\left(\frac{x}{\alpha}\right)^2$$

Therefore, $\text{LHS} = RHS$

Hence proved.

EXERCISE – 2: Basic Objective Questions

Section-A (Single Choice Questions)

1. Suppose $(r+1)^{th}$ term is independent of x.

We have $\left(2x+\frac{1}{3x^2}\right)^9$, then

$$T_{r+1} = {}^9C_r\,(2x)^{9-r}\left(\frac{1}{3x^2}\right)^r$$

$$= {}^9C_r\,2^{9-r}\,\frac{1}{3^r}.x^{9-3r}$$

This term is independent of x if $9-3r=0$

$\therefore\ r=3$.

Thus, 4^{th} term is independent of x.

2. The constant term

$$= {}^{10}C_4\left(\sqrt[3]{\frac{x}{3}}\right)^6\left(-\sqrt{\frac{3}{x}}\right)^4$$

$$= {}^{10}C_4\,\frac{1}{3^2}.3^2 = 210$$

3. Since, $T_{r+1} = {}^nC_r a^{n-r}x^r$ in expansion of $(a+x)^n$,

Therefore, $T_8 = {}^nC_7\,(2)^{n-7}\left(\frac{x}{3}\right)^7$

$$= {}^nC_7\,\frac{2^{n-7}}{3^7}x^7$$

and $T_9 = {}^nC_8\,(2)^{n-8}\left(\frac{x}{3}\right)^8$

$$= {}^nC_8\,\frac{2^{n-8}}{3^8}x^8$$

Therefore, ${}^nC_7\,\frac{2^{n-7}}{3^7} = {}^nC_8\,\frac{2^{n-8}}{3^8}$

it is given that

coefficient of x^7 = coefficient of x^8

$$\Rightarrow \frac{n!}{7!(n-7)!}\times\frac{8!(n-8)!}{n!} = \frac{2^{n-8}}{3^8}.\frac{3^7}{2^{n-7}}$$

$$\Rightarrow \frac{8}{n-7} = \frac{1}{6} \Rightarrow n = 55.$$

4. The $(r+1)^{th}$ term in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$ is given by

$$T_{r+1} = {}^{10}C_r\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\frac{3}{2x^2}\right)^r$$

$$= {}^{10}C_r\,\frac{x^{5-\left(\frac{r}{2}\right)}}{3^{5-\left(\frac{r}{2}\right)}}.\frac{3^r}{2^r x^{2r}}$$

$$= {}^{10}C_r\,\frac{3^{\left(\frac{3r}{2}\right)-5}}{2^r}x^{5-\left(\frac{5r}{2}\right)}$$

For T_{r+1} to be independent of x,

we must have, $5-\left(\frac{5r}{2}\right)=0$ or $r=2$.

Thus, the 3^{rd} term is independent of x and is equal to ${}^{10}C_2\,\frac{3^{3-5}}{2^2} = \frac{10\times 9}{2}\times\frac{3^{-2}}{4} = \frac{5}{4}$.

5. T_r is the r^{th} term in the expansion of $(1+x)^{101}$.

$T_r = {}^{101}C_{r-1}.(x)^{(r-1)}$

$\therefore \dfrac{T_{20}}{T_{19}} = \dfrac{{}^{101}C_{19}}{{}^{101}C_{18}}.\dfrac{x^{19}}{x^{18}} = \dfrac{{}^{101}C_{19}x}{{}^{101}C_{18}}$

$= \dfrac{\dfrac{101!}{(19!)(82!)}}{\dfrac{101!}{(18!)(83!)}}x = \dfrac{83x}{19}$.

6. Since co-efficient of 5^{th}, 6^{th} and 7^{th} terms are in A.P

$\therefore\ {}^nC_4 + {}^nC_6 = 2.{}^nC_5$

$\Rightarrow \dfrac{n!}{(n-4)!\times 4!} + \dfrac{n!}{(n-6)!\times 6!} = 2.\dfrac{n!}{(n-5)!\times 5!}$

$\Rightarrow 6.5 + (n-4)(n-5) = 2(n-4).6$

$\Rightarrow 30 + n^2 - 9n + 20 = 12n - 48$

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow n = 7, 14$

7. When exponent is n, then total number of terms are $n+1$. So, total number of terms in $(2+3x)^4 = 5$

Middle term is 3^{rd} term.

$\Rightarrow T_3 = {}^4C_2\,(2)^2.(3x)^2$

$= \dfrac{4\times 3\times 2\times 1}{2\times 1\times 2}\times 4\times 9x^2 = 216x^2$

$\therefore$ Coefficient of middle term is 216.

8. In the given expansion of $\left(\dfrac{x}{3} - \dfrac{2}{x^2}\right)^{10}$,

We have,

$T_r = {}^{10}C_{r-1}\left(\dfrac{x}{3}\right)^{10-r+1}\left(\dfrac{-2}{x^2}\right)^{r-1}$

$= \left[{}^{10}C_{r-1}\right]x^{13-r-2r}.(-2)^{r-1}\left(\dfrac{1}{3}\right)^{11-r}$

r^{th} term contains x^4 when
$13 - 3r = 4 \Rightarrow r = 3$.

9. In the expansion of $\left(x^4 + \dfrac{1}{x^3}\right)^{15}$,

let T_r be the r^{th} term

$T_r = {}^{15}C_{r-1}\left(x^4\right)^{15-r+1}\left(\dfrac{1}{x^3}\right)^{r-1}$

$= {}^{15}C_{r-1}\,x^{64-4r-3r+3} = {}^{15}C_{r-1}x^{67-7r}$

x^4 occurs in this term
$\Rightarrow 4 = 67 - 7r$.
$\Rightarrow 7r = 63 \Rightarrow r = 9$

10. In the expansion of $\left(2x + 3y^2\right)^5$,

We have, $T_r = {}^nC_{r-1}\,(2x)^{r-1}\left(3y^2\right)^{n-r+1}$

$T_4 = T_{3+1} = {}^5C_3\,(2x)^3\left(3y^2\right)^2$

$= \dfrac{5!}{3!2!}2^3.x^3.9y^4 = \dfrac{5.4}{2.1}\times 8\times 9\times x^3y^4$

$= 720x^3y^4$

$\therefore$ Coefficient of $x^3y^4 = 720$.

11. T_{r+1} in the expansion of $\left[ax^2 + \dfrac{1}{bx}\right]^{11}$ is

$= {}^{11}C_r\left(ax^2\right)^{11-r}\left(\dfrac{1}{bx}\right)^r$

$= {}^{11}C_r\,(a)^{11-r}(b)^{-r}(x)^{22-2r-r}$

For the coefficient of x^7, We have,
$22 - 3r = 7 \Rightarrow r = 5$

$\therefore$ Coefficient of x^7 is

${}^{11}C_5(a)^6(b)^{-5}$ ......(i)

Again T_{r+1} in the expansion of

$\left[ax-\frac{1}{bx^2}\right]^{11} = {}^{11}C_r\left(ax^2\right)^{11-r}\left(-\frac{1}{bx^2}\right)^r$

$= {}^{11}C_r(a)^{11-r}(-1)^r \times (b)^{-r}(x)^{-2r}(x)^{11-r}$

For the coefficient of x^{-7},

we have

$11-3r=-7=18 \Rightarrow r=6$

$\therefore$ coefficient of $x^{-7} = {}^{11}C_6 a^5 \times 1 \times (b)^{-6}$

By the given,

${}^{11}C_5(a)^6(b)^{-5} = {}^{11}C_6 a^5 \times (b)^{-6}$

$\Rightarrow ab=1$.

12. The expansion of

$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 +$

$\ldots..{}^{2n}C_n x^n + \ldots. + {}^{2n}C_{2n} x^{2n}$ .....(i)

And

$(1+x)^{2n-1} = {}^{2n-1}C_0 + {}^{2n-1}C_1 x + {}^{2n-1}C_2 x^2 \ldots..$

${}^{2n-1}C_n x^n + \ldots. + {}^{2n-1}C_{2n-1} x^{2n-1}$ ....(ii)

According to the given data and equations (i) and (ii), we can claim that

$A = {}^{2n}C_n$ and $B = {}^{2n-1}C_n$

$$\Rightarrow \frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{2n!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}}$$

$$\Rightarrow \frac{A}{B} = \frac{2n(2n-1)!}{n(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = 2$$

$\Rightarrow A = 2B$.

13. We have, $\frac{(3-2x)}{(1+3x)^3} = (3-2x)(1+3x)^{-3}$

$= (3-2x)\left[1-9x+\frac{(-3)(-4)}{2!}.9x^2+\frac{(-3)-(-4)(-5)}{3!}.27x^3+\ldots..\right]$

$\left[\text{Expanding } (1+3x)^{-3}\right]$

$= (3-2x)\left(1-9x+54x^2-270x^3+\ldots...\right)$

$\therefore$ Coefficient of $x^3 = -270\times 3 - 2\times 54$

$= -810-108 = -918$.

14. $\sqrt{5}\left[\left(\sqrt{5}+1\right)^{50} - \left(\sqrt{5}-1\right)^{50}\right]$

$= 2\sqrt{5}\left[{}^{50}C_1\left(\sqrt{5}\right)^{49} + {}^{50}C_3\left(\sqrt{5}\right)^{47} + \ldots..\right]$

$= 2\left[{}^{50}C_1\left(\sqrt{5}\right)^{50} + {}^{50}C_3\left(\sqrt{5}\right)^{48} + \ldots..\right]$

= A natural number.

15. $\left[(x+4y)^3(x-4y)^3\right]^2$

$= \left[\left(x^2-(4y)^2\right)\right]^6 = \left(x^2-16y^2\right)^6$

$\therefore$ Number of terms in the expansion is 7.

16. Let $(r+1)^{th}$ term be independent of x in

the expansion of $\left(\sqrt[6]{x}-\frac{1}{\sqrt[3]{x}}\right)^9$

$T_{r+1} = {}^9C_r\left(\sqrt[6]{x}\right)^{9-r}\left(-\frac{1}{\sqrt[3]{x}}\right)^r$

$= {}^9C_r(-1)^r.x^{\frac{9-r}{6}-\frac{r}{3}} = {}^9C_r.x^{\left(\frac{9-3r}{6}\right)}$

Now, $\frac{9-3r}{6} = 0 \Rightarrow r=3$;

Thus, term independent of $x = -{}^9C_3$.

17. We know that the coefficient in a binomial expansion is obtained by replacing each variable by unit in the given expression. Therefore, the sum of the coefficients in $(a+b)^n = (1+1)^n$

$\Rightarrow 2^n = 4096 = 2^{12} \Rightarrow n = 12$

The greatest coefficient = coefficient of middle term.

So, middle term $= T_7$

Coeff of $T_7 = {}^{12}C_6 = \dfrac{12!}{6!6!} = 924$.

18. Suppose x^{-7} occurs in $(r+1)^{th}$ term.

we have $T_{r+1} = {}^nC_r x^{n-r} a^r$ in $(x-a)^n$.

In the given expansion $\left[ax - \dfrac{1}{bx^2}\right]^{11}$,

we have $n = 1,\ x = ax,\ a = \dfrac{-1}{bx^2}$

$\therefore T_{r+1} = {}^{11}C_r (ax)^{11-r} \left(\dfrac{-1}{bx^2}\right)^r$

$= {}^{11}C_r a^{11-r} b^{-r} x^{11-3r} (-1)^r$

This term contains x^{-7} if $11 - 3r = -7$

$\Rightarrow r = 6$

Therefore, coefficient of x^{-7} is

$${}^{11}C_6 (a)^5 \left(\frac{-1}{b}\right)^6 = \frac{462}{b^6} a^5.$$

19. The given expression,

$$\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\left(1+\frac{C_3}{C_2}\right)\ldots\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$$

$$= \left(1+\frac{n}{1}\right)\left(1+\frac{n-1}{2}\right)\left(1+\frac{n-2}{3}\right)\ldots\left(1+\frac{1}{n}\right)$$

$$= \frac{(n+1)^n}{n!}.$$

20. In the expansion of $(3+7x)^{29}$

We have,

$T_{r+1} = {}^{29}C_r . 3^{29-r} . (7x)^r$

$= \left({}^{29}C_r . 3^{29-r} . 7^r\right) x^r$

$\therefore a_r$ = coefficient of $(r+1)^{th}$ term

$= {}^{29}C_r . 3^{29-r} . 7^r$

Now, $a_r = a_{r-1}$

$\Rightarrow {}^{29}C_r . 3^{29-r} . 7^r = {}^{29}C_{r-1} . 3^{30-r} . 7^{r-1}$

$$\Rightarrow \frac{{}^{29}C_r}{{}^{29}C_{r-1}} = \frac{3}{7} \Rightarrow \frac{30-r}{r} = \frac{3}{7} \Rightarrow r = 21.$$

Section–B (Case Study Questions)

Case Study-1

21. (i). The $(r+1)^{th}$ term in the expansion of $\left(2+\dfrac{x}{3}\right)^n$ is given by

$$T_{r+1} = {}^nC_r 2^{n-r} \left(\frac{x}{3}\right)^r$$

$$= {}^nC_r \left(\frac{2^{n-r}}{3^r}\right) x^r$$

According to the given condition

$${}^nC_7 \left(\frac{2^{n-7}}{3^7}\right) = {}^nC_8 \left(\frac{2^{n-8}}{3^8}\right)$$

$$\Rightarrow \frac{{}^nC_7}{{}^nC_8} = \frac{2^{n-8}}{3^8} . \frac{3^7}{2^{n-7}}$$

$$\Rightarrow \frac{n!}{7!(n-7)!} \frac{8!(n-8)!}{n!} = \frac{1}{6}$$

$$\Rightarrow \frac{8}{n-7} = \frac{1}{6}$$

$\Rightarrow 48 = n - 7$

$\Rightarrow n = 55$

(ii). The $(r+1)^{th}$ term in the expansion of

$\left(x^3-\frac{1}{x^2}\right)^{15}$ is

$$T_{r+1}={}^{15}C_r\left(x^3\right)^{15-r}\left(\frac{-1}{x^2}\right)^r$$

$$={}^{15}C_r\left(-1\right)^r x^{45-5r}$$

To obtain constant term,
we get
$45-5r=0$
$\Rightarrow r=9$
$\therefore$ Co-efficient of constant term is
${}^{15}C_9\left(-1\right)^9=-{}^{15}C_9$

(iii). The middle term of $\left(\frac{a}{3}+3\right)^8$ is

$\left(\frac{8}{2}+1\right)^{th}$ term or 5^{th} term

We have,

$$T_5=T_{4+1}={}^8C_4\left(\frac{a}{3}\right)^4 3^4$$

$$=70a^4$$

We are given $T_5=1120$

$\Rightarrow 70a^4=1120$

$\Rightarrow a^4=16$

$\Rightarrow a^4=2^4$

$\therefore\ a=\pm 2$

(iv). The Middle term in the expansion of $(1+\alpha x)^4$ is

$T_3={}^4C_2\left(\alpha x\right)^2=6\alpha^2x^2$ and

the middle term in the expansion of $(1-\alpha x)^6$ is $T_4={}^6C_3\left(-\alpha x\right)^3=-20\alpha^3x^3$

We are given, $6\alpha^2=-20\alpha^3$

$\Rightarrow \alpha=0$ or $\alpha=\frac{-3}{10}$

Case Study-2

22. (i). Let term be independent of $'x'$ in the expression $\left(3x^2-\frac{1}{2x^3}\right)^{10}$.

Now, $T_{r+1}={}^{10}C_r\left(3x^2\right)^{10-r}\left(-\frac{1}{2x^3}\right)^r$

$$={}^{10}C_r\cdot 3^{10-r}\cdot\left(-\frac{1}{2}\right)^r\cdot x^{20-5r}$$

The term will be independent of $'x'$ if
$20-5r=0\Rightarrow r=4$
$\therefore 5^{th}$ term is independent of $'x'$

$$T_5={}^{10}C_4\cdot 3^{10-4}\cdot\left(-\frac{1}{2}\right)^4.$$

$$=\frac{10\times9\times8\times7}{4\times3\times2\times1}\times\frac{729}{16}=\frac{76545}{8}$$

when it is multiplied by 8, we get 76545.

(ii). Since $n=10$, which is even, so

$\left(\frac{10}{2}+1\right)^{th}$ term i.e. 6^{th} term is middle term.

Hence, the middle term

$$=T_6={}^{10}C_5\cdot 3^{10-5}\cdot\left(-\frac{1}{2}\right)^5\cdot x^{20-5(5)}$$

$$=-{}^{10}C_5\times 3^5\times\frac{1}{32}\times x^{-5}$$

$$=-\frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}\times\frac{243}{32x^5}=\frac{-15309}{8x^5}.$$

(iii) $T_{r+1}={}^{10}C_r\cdot 3^{10-r}\cdot\left(-\frac{1}{2}\right)^r\cdot x^{20-5r}$

for $x^{10}, 20-5r=10\Rightarrow r=2$

$\therefore 3^{rd}$ term is the term with x^{10}

Now $T_{r+1}={}^{10}C_r\cdot 3^{10-r}\cdot\left(-\frac{1}{2}\right)^r\cdot x^{20-5r}$

$$={}^{10}C_2\cdot 3^8\cdot\frac{1}{4}=\frac{10\times9}{2}\times\frac{3^8}{4}=\frac{5\times3^{10}}{4}$$

(iv) The 5^{th} term from the end of $\left(3x^2-\frac{1}{2x^3}\right)^{10}$ is equal to 5^{th} term from the beginning of $\left(-\frac{1}{2x^3}+3x^2\right)^{10}$

i.e. $T_5={}^{10}C_4\cdot\left(-\frac{1}{2x^3}\right)^{10-4}\cdot\left(3x^2\right)^4$

$={}^{10}C_4\cdot\frac{1}{2^6}\cdot 3^4\cdot x^{-18+8}$

$={}^{10}C_4\times\frac{81}{64}\times\frac{1}{x^{10}}=\frac{8505}{32x^{10}}$.

Case Study-3

23. (i). Given expansion is $\left(2x^2-\frac{1}{x}\right)^8$.

The general term of an expansion is

$T_{r+1}={}^nC_r x^{n-r}y^r$

$\therefore$ 4^{th} term is $T_4=T_{3+1}$

$\Rightarrow T_{3+1}={}^8C_3\left(2x^2\right)^{8-3}\left(\frac{-1}{x}\right)^3$

$={}^8C_3\left(2x^2\right)^5(-1)^3x^{-3}$

$=\frac{8!}{5!3!}2^5.x^{10-3}.(-1)^3$

$=-\frac{8\times7\times6\times5!}{5!\times3!}.(32)x^7$

$=-1792x^7$

(ii). In the expansion of $(1-2x)^7$,

the middle terms are $\left(\frac{7+1}{2}\right)^{th}$ term and $\left(\frac{7+1}{2}+1\right)^{th}$ terms,

i.e 4^{th} and 5^{th} terms are middle terms of $(1-2x)^7$.

$T_4=T_{3+1}={}^7C_3(1)^{7-3}(-2x)^3$

$=\frac{7!}{4!\times3!}(-8)\left(x^3\right)$

$=\frac{7\times6\times5\times4!}{4!\times3!}(-8)x^3=-280x^3$

And $T_5=T_{4+1}={}^7C_4(1)^{7-4}(-2x)^4$

$=\frac{7!}{4!\times3!}(16)x^4$

$=\frac{7\times6\times5\times4!}{4!\times3!}(16)x^4$

$=560x^4$

$\therefore$ The Sum of the co-efficient of the middle terms $=-280+560$

$=280$

(iii). Let $(r+1)^{th}$ term be the term independent of x in the expansion of $\left(2x^2-\frac{1}{x}\right)^8$

$\therefore T_{r+1}={}^8C_r\left(2x^2\right)^{8-r}\left(-\frac{1}{x}\right)^r$

$={}^8C_r 2^{8-r}x^{16-2r-r}$

$={}^8C_r 2^{8-r}x^{16-3r}$

For the term to be independent of x,

$16-3r=0$

$r=\frac{16}{3}$

Since, r is a fraction, there is no term independent of x in the given term.

(iv). Given expansion is $(1-2x)^7$

The 6^{th} term from the end of $(1-2x)^7=6^{th}$ term from the beginning of $(2x-1)^7$

$\therefore T_6=T_{5+1}={}^7C_5(2x)^{7-5}(-1)^5$

$={}^7C_5(2x)^2(-1)$

$=-{}^7C_5.4x^2$

Section–C (Assertion/Reason type Questions)

24. Given that, $(1+ax)^n = 1+8x+24x^2+....$

$\Rightarrow 1+\frac{n}{1}ax+\frac{n(n-1)}{1.2}a^2x^2+.... = 1+8x+24x^2+....$

On comparing the coefficient of x, x^2, we get

$na = 8, \quad \frac{n(n-1)}{2}a^2 = 24$

$\Rightarrow na(na-a) = 48 \Rightarrow 8(8-a) = 48$

$\Rightarrow 8-a = 6$

$\Rightarrow a = 2$

$\therefore n\times 2 = 8$

$\Rightarrow n = 4$

25. We have,

$\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)....\left(1+\frac{C_n}{C_{n-1}}\right)$

$=\left(1+\frac{n}{1}\right)\left[1+\frac{\frac{n(n-1)}{2!}}{n}\right]....\left(1+\frac{1}{n}\right)$

$=\frac{(1+n)}{1}.\frac{(1+n)}{2}.\frac{(1+n)}{3}....\frac{(1+n)}{n}=\frac{(1+n)^n}{n!}$.

26. There are $(n+1)$ terms in the expansion of $(x+a)^n$. Observing the terms, we can say that the first term from the end is the last term, i.e., $(n+1)^{th}$ term of the expansion and $n+1=(n+1)-(1-1)$ The second term from the end is the n^{th} term of the expansion and $n=(n+1)-(2-1)$

The third term from the end is the $(n-1)^{th}$ term of the expansion and $n-1=(n+1)-(3-1)$, and so on. Thus, r^{th} term from the end will be term number $(n+1)-(r-1)=(n-r+2)$ of the expansion and the $(n-r+2)^{th}$ term is ${}^nC_{n-r+1}x^{r-1}a^{n-r+1}$

27. In the expansion of $(x+2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

28. Assertion: $(x+2y)^9$

$n=9,\ a=2y$

$\therefore T_{r+1} = {}^9C_r.x^{9-r}.(2y)^r$

$= {}^9C_r.2^r.x^{9-r}.y^r$.

29. Both are correct, and Reason is the correct explanation.

30. Assertion is correct. Reason is false.

Total number of terms $=\left(\frac{n}{2}+1\right)=5+1=6$